EXTREMAL THEORY FOR LONG RANGE DEPENDENT INFINITELY DIVISIBLE PROCESSES

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Abstract. We prove limit theorems of an entirely new type for certain long memory regularly varying stationary infinitely divisible random processes. These theorems involve multiple phase transitions governed by how long the memory is. Apart from one regime, our results exhibit limits that are not among the classical extreme value distributions. Restricted to the one-dimensional case, the distributions we obtain interpolate, in the appropriate parameter range, the α-Fréchet distribution and the skewed α-stable distribution. In general, the limit is a new family of stationary and self-similar random sup-measures with parameters $\alpha \in (0,\infty)$ and $\beta \in (0,1)$, with representations based on intersections of independent β-stable regenerative sets. The tail of the limit random sup-measure on each interval with finite positive length is regularly varying with index $-\alpha$. The intriguing structure of these random sup-measures is due to intersections of independent β-stable regenerative sets and the fact that the number of such sets intersecting simultaneously increases to infinity as β increases to one. The results in this paper extend substantially previous investigations where only $\alpha \in (0,2)$ and $\beta \in (0,1/2)$ have been considered.

1. Introduction

Given a stationary process $(X_n)_{n\in\mathbb{N}}$, we are interested in the asymptotic behavior of the maximum

$$M_n := \max_{i=1,\dots,n} X_i.$$

After appropriate normalization, what distributions may arise in the limit? This is a classical question in probability theory with a very long history. In the case that $(X_n)_{n\in\mathbb{N}}$ is a sequence of independent and identically distributed (i.i.d.) random variables, all possible limits of the weak convergence in the form of

$$\frac{M_n - a_n}{b_n} \Rightarrow Z \tag{1}$$

have been known since Fisher and Tippett (1928) and Gnedenko (1943): these form the family of extreme-value distributions, consisting of Fréchet, Gumbel and

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Weibull types. Furthermore, the functional extremal limit theorem in the form of

$$\left(\frac{M_{\lfloor nt\rfloor}-a_n}{b_n}\right)_{t\geq 0}\Rightarrow (Z(t))_{t\geq 0} \tag{2}$$

in an appropriate topological space has also been known since Dwass (1964) and Lamperti (1964). The limit process Z, when non-degenerate, is known as the *extremal process.*

If a stationary process $(X_n)_{n\in\mathbb{N}}$ is not a sequence of i.i.d. random variables, the extremes can cluster, and this can affect the extremal limit theorems for such processes. Research along this line has started since the 60s. A common feature of many results in the literature on this topic is the important role of the so-called *extremal index* $\theta\in(0,1]$. When this index exists, it affects the limit theorems through the fact that, asymptotically, the limit law of M_n is the same as that of $\widetilde{M}_{\lfloor\theta n\rfloor}$, the maximum of $\lfloor\theta n\rfloor$ i.i.d. copies of X_1, when one uses the same normalization in both cases. This reflects the following picture of extremes of such processes: extreme values of the process occur in finite random clusters, the smaller θ indicates larger, on average, cluster size. It is also worth noting that for all $\theta\in(0,1]$, the order of the normalization and the limit laws in (1) and (2) are the same as in the i.i.d. case. Therefore, one can view processes with extremal index $\theta\in(0,1]$ as having, in the appropriate sense, *short memory* (the reasons for this terminology can be found in Samorodnitsky (2016)). Standard references for extreme value theory on i.i.d. and weakly dependent sequences include Leadbetter et al. (1983); Resnick (1987); de Haan and Ferreira (2006). Point-process techniques are fundamental and powerful when investigating such problems.

There are situations that for the limit theorems of the types (1) and (2) to hold, the normalization needs to be of a different order, and even the limit may be different, from the short memory case. We refer to the dependence in such examples as strong or *long range dependence.* See the recent monograph Samorodnitsky (2016) for more background and recent developments on long range dependence in terms of limit theorems (not necessarily extremal ones). The first example of long range dependence in extreme value theory is for stationary Gaussian processes: Mittal and Ylvisaker (1975) showed that when the correlation r_n satisfies $\lim_{n\to\infty} r_n\log n=\gamma\in(0,\infty)$, the limit law of M_n is Gumbel convoluted with a Gaussian distribution, in contrast to the case of $\lim_{n\to\infty} r_n\log n=0$ where the Gumbel distribution arises in the limit, due to Berman (1964). However, very few examples of extremes of stationary non-Gaussian processes with long range dependence have been discovered since then. One of the known examples is important for us in this paper and we will discuss it below.

A fundamental work is due to O'Brien et al. (1990) who, in the process of identifying all possible limits of extremes of a sequence of stationary random variables, pointed out that a more natural and revealing way to investigate extremes is via the *random sup-measures.* In this framework, for each n one investigates the random sup-measure M_n in the form of

$$M_n(B):=\max_{k\in nB\cap\mathbb{N}} X_k, B\subset\mathbb{R}_+,$$

in an appropriate topological space. Then, a limit theorem for M_n entails at least the finite-dimensional convergence part in a functional extremal limit theorem as in (2) when restricted to all B in the form of $B=[0,t], t\geq 0$. O'Brien et al.

(1990) showed that all possible random sup-measures η on $[0,\infty)$ arising as limits starting from a stationary process $(X_n)_{n\in\mathbb{N}}$ are, up to affine transforms, stationary and self-similar, in the sense that

$$\eta(\cdot) \overset{d}{=} \eta(\cdot + b), b > 0 \quad \text{ and } \quad \eta(a\cdot) \overset{d}{=} a^H \eta(\cdot), a > 0$$

for some $H > 0$. They also provided examples of such random sup-measures. However, the investigation of O'Brien et al. (1990) does not directly help in understanding extremal limit theorems under long range dependence.

In this paper, we investigate the extremes of a general class of stationary infinitely divisible processes whose law is linked to the law of a certain null-recurrent Markov chain. Two crucial numerical parameters impact the properties of such infinitely divisible processes: $\alpha \in (0,\infty)$ and $\beta \in (0,1)$: the parameter α corresponds to the regular variation index of the tail of the marginal distribution, and β determines the rate of the recurrence of the underlying Markov chain (the larger the β, the faster the rate) and, as a result, plays an important role in determining the memory of the infinitely divisible process. The extremes of symmetric α-stable processes in this class have been first investigated in Samorodnitsky (2004), who showed that when $\beta \in (0,1/2)$, the partial maxima converge weakly to the Fréchet distribution, although under the normalization $b_n = n^{(1-\beta)/\alpha}$ instead of $n^{1/\alpha}$ used in the i.i.d. case. (Since infinitely divisible processes we are considering are heavy-tailed, we take the shift $a_n = 0$ in all extremal limit theorems.) The different order of normalization already indicates long range dependence of the process. Furthermore, it was pointed out in the same paper that when $\beta \in (1/2,1)$, the dependence was so strong that the partial maxima were likely not to converge to the Fréchet distribution, but an alternative limit distribution was not described.

Further studies of the extrema of this class of processes have appeared more recently, still in the symmetric α-stable case, with $\beta \in (0,1/2)$ (though in a different notation). In Owada and Samorodnitsky (2015b), it was shown that the limit in the functional extremal theorem as in (2) is, up to a multiplicative constant, a time-changed extremal process,

$$\left(Z_\alpha(t^{1-\beta})\right)_{t\geq 0},$$

where $(Z_\alpha(t))_{t\geq 0}$ is the extremal process for a sequence of i.i.d. random variables with tail index α (the α-Fréchet extremal process). Subsequently, Lacaux and Samorodnitsky (2016), established a limit theorem in the framework of convergence of random sup-measures, and, up to a multiplicative constant, the limit random sup-measure can be represented as

$$\eta(\cdot) = \bigvee_{j=1}^{\infty} U_j^{(\alpha)} \mathbf{1}_{\left\{\left(V_j^{(\beta)} + R_j^{(\beta)}\right) \cap \cdot \neq \emptyset\right\}}, \tag{3}$$

where $(U_j^{(\alpha)}, V_j^{(\beta)}, R_j^{(\beta)})_{j\in\mathbb{N}}$ is a measurable enumeration of the points of a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{F}(\mathbb{R}_+)$ with intensity $\alpha u^{-\alpha-1}du(1-\beta)v^{-\beta}dvdP_\beta$. Here $\mathcal{F}(\mathbb{R}_+)$ is the space of closed subsets of $\mathbb{R}_+$ equipped with Fell topology, and P_β is the law of a β-stable regenerative set, the closure of the range of a β-stable subordinator, on $\mathcal{F}(\mathbb{R}_+)$. Then

$$(\eta([0,t]))_{t\geq 0} \overset{d}{=} \left(Z_\alpha(t^{1-\beta})\right)_{t\geq 0},$$

but the random sup-measure reveals more structure than the time-changed extremal process.

In this paper we fill the gaps left in the previous studies. First of all, we move away from the assumption of stability to a more general class of stationary infinitely divisible processes. This allows us to remove the restriction of $\alpha \in (0,2)$ in our limit theorems. Much more importantly, we remove the assumption $\beta \in (0,1/2)$. This allows us to consider the extrema of processes whose memory is very long. Our results confirm that the Fréchet limits obtained in Samorodnitsky (2004) and the subsequent publications disappear when $\beta \in (1/2,1)$. In fact, entirely new limits appear. Even the one-dimensional distributions we obtain as marginal limits have not, to the best of our knowledge, been previously described. The limiting random sup-measure turns out to be uniquely determined by the random upper-semi-continuous function

$$\eta^{\alpha,\beta}(t) := \sum_{j=1}^{\infty} U_j^{(\alpha)} \mathbf{1}_{\left\{t \in V_j^{(\beta)} + R_j^{(\beta)}\right\}}, t \geq 0,$$

with $(U_j^{(\alpha)}, V_j^{(\beta)}, R_j^{(\beta)})_{j\in\mathbb{N}}$ as before. When $\beta \in (0,1/2]$, this is the same random sup-measure as the one in (3), as independent β-stable regenerative sets do not intersect for such a β. For $\beta > 1/2$, however, eventual intersections occur almost surely, and the larger the β becomes, more independent regenerative sets can intersect at the same time. As Section 3 below shows, for every $\alpha \in (0,\infty)$, $(\eta^{\alpha,\beta})_{\beta\in(0,1)}$ forms a family of random sup-measures corresponding to the full range of dependence: from independence ($\beta \downarrow 0$) to complete dependence ($\beta \uparrow 1$). Importantly, if $\alpha \in (0,1)$, the the marginal distributions, for example those of $\eta^{\alpha,\beta}([0,1])$, form a family of distributions that interpolate between the α-Fréchet distribution (resulting when $\beta \in (0,1/2]$) and the totally skewed to the right α-stable distribution as $\beta \uparrow 1$.

The paper is organized as follows. In Section 2 we present background information on random closed sets and random sup-measures. In Section 3 we introduce and investigate the limiting random sup-measure. The stationary infinitely divisible process with long range dependence whose extremes we study is introduced in Section 4, and a limit theorem for these extremes in the context of random sup-measures is proved in Section 5.

2. Random closed sets and sup-measures

We first provide background on random closed sets. Our main reference is Molchanov (2005). Let $\mathcal{F}(E)$ denote the space of all closed subsets of an interval $E \subset \mathbb{R}$. In this paper we only work with $E = [0,1]$ and $E = [0,\infty)$. The space $\mathcal{F} = \mathcal{F}(E)$ is equipped with the Fell topology generated by

$$\mathcal{F}_G := \{F \in \mathcal{F} : F \cap G \neq \emptyset\} \text{ for all } G \in \mathcal{G},$$

where $\mathcal{G} = \mathcal{G}(E)$ the collection of all open subsets of E, and

$$\mathcal{F}^K := \{F \in \mathcal{F} : F \cap K = \emptyset\} \text{ for all } K \in \mathcal{K},$$

whre $\mathcal{K} = \mathcal{K}(E)$ is the collection of all compact subsets of E. This topology is metrizable, and $\mathcal{F}(E)$ is compact under it. If $\mathcal{F}$ is equipped with the Borel σ-algebra $\mathcal{B}(\mathcal{F})$ induced by the Fell topology, a random closed set is a measurable mapping from a probability space to $(\mathcal{F}, \mathcal{B}(\mathcal{F}))$. Given random closed sets $(R_n)_{n\in\mathbb{N}}$ and R, a sufficient condition for weak convergence $R_n \Rightarrow R$ is

$$\lim_{n\to\infty} \mathbb{P}(R_n \cap A \neq \emptyset) = \mathbb{P}(R \cap A = \emptyset), \text{ for all } A \in \mathcal{A} \cap \mathfrak{S}_R,$$

where $\mathcal{A}$ is the collection of all finite unions of open intervals, and $\mathfrak{S}_R$ is the collection of all continuity sets of R: the collection of relatively compact Borel sets B such that $\mathbb{P}(R \cap \overline{B} \neq \emptyset) = \mathbb{P}(R \cap B^o \neq \emptyset)$. See Molchanov (2005), Corollary 1.6.9 (the collection $\mathcal{A}$ is called a *separating class*).

We proceed with background on sup-measures and upper-semi-continuous functions. Our main reference is O'Brien et al. (1990). See also Molchanov and Strokorb (2016) and Sabourin and Segers (2016) for some recent developments. Let E be as above, and $\mathcal{G} = \mathcal{G}(E)$ the collection of open subsets of E. A map $m : \mathcal{G} \to [0, \infty]$ is a sup-measure, if

$$m\left(\bigcup_\alpha G_\alpha\right) = \sup_\alpha m(G_\alpha)$$

for all arbitrary collections of open sets $(G_\alpha)_\alpha$. Given a sup-measure m, its sup-derivative, denoted by $d^\vee m : E \to [0, \infty]$, is defined as

$$d^\vee m(t) := \inf_{G\ni t} m(G), t \in E.$$

The sup-derivative of a sup-measure is an upper-semi-continuous function, that is a function f such that $\{f < t\}$ is open for all $t > 0$. Given an $[0, \infty]$-valued upper-semi-continuous function f, the sup-integral $i^\vee f : \mathcal{G} \to [0, \infty]$ is defined as

$$i^\vee f(G) := \sup_{t\in G} f(t), G \in \mathcal{G},$$

with $i^\vee f(\emptyset) = 0$ by convention. The sup-integral is a sup-measure. Let $\mathrm{SM} = \mathrm{SM}(E)$ and $\mathrm{USC} = \mathrm{USC}(E)$ denote the spaces of all sup-measures on E and all $[0, \infty]$-valued upper-semi-continuous functions on E, respectively. It turns out that $d^\vee$ is a bijection between SM and USC, and $i^\vee$ is its inverse. Every $m \in \mathrm{SM}$ has a canonical extension to all subsets of E, given by

$$m(B) = \sup_{t\in B} (d^\vee m)(t), B \subset E.$$

The space SM is equipped with the so-called sup-vague topology. In this topology, $m_n \to m$ if and only if

$$\limsup_{n\to\infty} m_n(K) \le m(K) \text{ for all } K \in \mathcal{K}$$

and

$$\liminf_{n\to\infty} m_n(G) \ge m(G) \text{ for all } G \in \mathcal{G}.$$

This topology is metrizable and the space SM is compact in this topology. The sup-vague topology on the space USC is then induced by the bijection $d^\vee$, so the convergence of

$$m_n \to m \text{ in SM} \quad \text{and} \quad d^\vee m_n \to d^\vee m \text{ in USC}$$

are equivalent.

A random sup-measure is a random element in $(\mathrm{SM}, \mathcal{B}(\mathrm{SM}))$ with $\mathcal{B}(\mathrm{SM})$ the Borel σ-algebra induced by the sup-vague topology. A random upper-semi-continuous function is defined similarly. We will introduce the limiting random sup-measures in our limit theorem through their corresponding random upper-semi-continuous functions. When proving weak convergence for random sup-measures we will utilize the following fact: given random sup-measures $(\eta_n)_{n\in\mathbb{N}}$ and η, weak convergence $\eta_n \Rightarrow \eta$ in S is equivalent to the finite-dimensional convergence

$$(\eta_n(I_1),\ldots,\eta_n(I_m)) \Rightarrow (\eta(I_1),\ldots,\eta(I_m))$$

for all $m\in\mathbb{N}$ and all open and η-continuity intervals $I_1,\ldots,I_m$ (I is η-continuity if $\eta(I)=\eta(\bar{I})$ with probability one). See O'Brien et al. (1990), Theorem 3.2.

3. A new family of random sup-measures

Recall that for $\beta\in(0,1)$, a β-stable regenerative set is the closure of the range of a strictly β-stable subordinator, viewed as a random closed set in $\mathcal{F}(\mathbb{R}_+)$, and it has Hausdorff dimension β almost surely; see for example Bertoin (1999b). We need a result on intersections of independent stable regenerative sets presented below. A number of similar results can be found in literature, see for example Hawkes (1977), Fitzsimmons et al. (1985) and Bertoin (1999a). We could not however find the exact formulation needed, so we included a short proof.

Lemma 1. *Consider $v_1,v_2\in\mathbb{R}_+$, $v_1\neq v_2$ and $\beta_1,\beta_2\in(0,1)$. Let $R_1^{(\beta_1)}$ and $R_2^{(\beta_2)}$ be two independent stable regenerative sets with parameter β_1 and β_2 respectively. Then,*

$$\mathbb{P}\left(\left(v_1+R_1^{(\beta_1)}\right)\cap\left(v_2+R_2^{(\beta_2)}\right)\neq\emptyset\right)\in\{0,1\}. \tag{4}$$

The probability equals one, if and only if $\beta_{1,2}:=\beta_1+\beta_2-1\in(0,1)$, and in this case, the intersection has the law of a shifted $\beta_{1,2}$-stable regenerative set, i.e. a random element in $\mathcal{F}(\mathbb{R}_+)$ with a representation

$$V+R^{(\beta_{1,2})},$$

where $R^{(\beta_{1,2})}$ is a $\beta_{1,2}$-stable regenerative set, and $V>\max(v_1,v_2)$ is a random variable independent of $R^{(\beta_{1,2})}$.

Proof. We may and will assume that $v_1>v_2=0$, and drop the subscript in v_1. For $x>0$ and $i=1,2$ let $B_{x,i}$ be the overshoot of the point x by a strictly β_i-stable subordinator, $i=1,2$; that is,

$$B_{x,i}\stackrel{d}{=}\min\left(R_i^{(\beta_i)}\cap[x,\infty)\right)-x,\quad x\geq 0.$$

Define a sequence of positive random variables $A_0,A_1,\ldots$ by $A_0=v$, $A_{2n+1}=B^{(2n+1)}_{A_{2n},2}$, $n=0,1,2,\ldots$, $A_{2n}=B^{(2n)}_{A_{2n-1},1}$, $n=1,2,\ldots$, where different superscripts correspond to overshoots by independent subordinators. Then, by the strong Markov property, the probability of a nonempty intersection in (4) is simply

$$\mathbb{P}\left(\sum_{n=0}^{\infty}A_n<\infty\right). \tag{5}$$

The overshoot $B_{x,i}$ has the density given by

$$f_x(y)=c(\beta_i)x^{\beta_i}(x+y)^{-1}y^{-\beta_i},\ y>0, \tag{6}$$

where $c(\beta_i) = \sin(\pi\beta_i)/\pi$ (see e.g. Kyprianou (2006), Exercise 5.8.) This implies that $B_{x,i} \stackrel{d}{=} xB_{1,i}$ for $x > 0$. Grouping the terms together, we see that the probability in (5) is equal to

$$\mathbb{P}\left(\sum_{n=1}^{\infty}\prod_{j=1}^{n} C_j < \infty\right),$$

where $C_1, C_2, \ldots$ are i.i.d. random variables with $C_1 \stackrel{d}{=} B^{(1)}_{1,1}B^{(2)}_{1,2}$. An immediate conclusion is that the

$$\mathbb{P}\left(\sum_{n=0}^{\infty} A_n < \infty\right) = \begin{cases} 1 & \text{if } \mathbb{E}\log C_1 < 0, \\ 0 & \text{if } \mathbb{E}\log C_1 \geq 0\,. \end{cases}$$

However, by (6), after some elementary manipulations of the integrals, we have

$$\mathbb{E}\log C_1 = c(\beta_1)\int_0^\infty \frac{y^{-\beta_1}\log y}{1+y}\,dy + c(\beta_2)\int_0^\infty \frac{y^{-\beta_2}\log y}{1+y}\,dy = \varphi(\beta_1) - \varphi(1-\beta_2)$$

with

$$\varphi(\beta) = \left(\int_0^\infty \frac{y^{-\beta}\log y}{1+y}\,dy\right)\Big/\left(\int_0^\infty \frac{y^{-\beta}}{1+y}\,dy\right).$$

So if $\beta_2 = 1-\beta_1$, $\mathbb{E}\log C_1 = 0$, and it is enough to prove that the function $\varphi(\beta)$ is strictly decreasing in $\beta \in (0,1)$. To see this,

$$\begin{aligned}\varphi'(\beta_1) &= \left(\int \frac{y^{-\beta_1}}{1+y}\,dy\right)^{-2}\left[\left(\int \frac{y^{-\beta_1}\log y}{1+y}\,dy\right)^2 - \int \frac{y^{-\beta_1}(\log y)^2}{1+y}\,dy\int \frac{y^{-\beta_1}}{1+y}\,dy\right] \\ &= -\mathrm{Var}\big(\log B_{1,1}\big) < 0\,.\end{aligned}$$

This proves (4) together with the criterion for the value of 1. Finally, by the strong Markov property of the stable regenerative sets, if $\beta_1+\beta_2 > 1$, then

$$\left(v + R_1^{(\beta_1)}\right) \cap R_2^{(\beta_2)} \stackrel{d}{=} \sum_{n=0}^{\infty} A_n + \left(R_1^{(\beta_1)} \cap R_2^{(\beta_2)}\right),$$

where on the right hand side, the series is independent of the stable regenerative sets. Since it has been shown by Hawkes (1977) that

$$R_1^{(\beta_1)} \cap R_2^{(\beta_2)} \stackrel{d}{=} R^{(\beta_1+\beta_2-1)},$$

the proof of the lemma is complete. $\square$

We now proceed with defining a new class of random sup-measures, by first identifying the underlying random upper-semi-continuous function. From now on, $\beta \in (0,1)$ and $\alpha > 0$ are fixed parameters. Consider a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{F}(\mathbb{R}_+)$ with mean measure

$$\alpha u^{-(1+\alpha)}\,du(1-\beta)v^{-\beta}\,dv\,dP_{R^{(\beta)}},$$

where $P_{R^{(\beta)}}$ is the law of the β-stable regenerative set. We let $(U_j^{(\alpha)}, V_j^{(\beta)}, R_j^{(\beta)})_{j\in\mathbb{N}}$ denote a measurable enumeration of the points of the point process, and

$$\widetilde{R}_j^{(\beta)} := V_j^{(\beta)} + R_j^{(\beta)},\ j \in \mathbb{N}$$

denote the random closed sets $R_j^{(\beta)}$ shifted by $V_j^{(\beta)}$. These are again random closed sets.

Introduce the intersection of such random closed sets with indices from $S \subseteq \mathbb{N}$ by

$$I_S := \bigcap_{j \in S} \widetilde{R}_j^{(\beta)}, S \neq \emptyset \quad \text{and} \quad I_\emptyset := \mathbb{R}_+. \tag{7}$$

Let

$$\ell_\beta := \max\left\{\ell \in \mathbb{N} : \ell < \frac{1}{1-\beta}\right\} \in \mathbb{N}.$$

By Lemma 1, we know that

$$I_S \begin{cases} \neq \emptyset & \text{a.s. if } |S| \leq \ell_\beta \\ = \emptyset & \text{a.s. if } |S| > \ell_\beta. \end{cases}$$

Furthermore when $|S| \leq \ell_\beta$, I_S is a randomly shifted stable regenerative set with parameter $\beta_{|S|}$, where

$$\beta_\ell := \ell\beta - (\ell - 1) \in (0,1) \quad \text{for all} \quad \ell = 1, \ldots, \ell_\beta.$$

Let

$$\eta^{\alpha,\beta}(t) := \sum_{j=1}^\infty U_j^{(\alpha)} \mathbf{1}_{\left\{t \in \widetilde{R}_j^{(\beta)}\right\}}, \, t \in \mathbb{R}_+. \tag{8}$$

Since a stable regenerative set does not hit fixed points, for every t, $\eta^{\alpha,\beta}(t) = 0$ almost surely. Furthermore, on an event of probability 1, every t belongs to at most ℓ_β different $\widetilde{R}_j^{(\beta)}$, and thus $\eta^{\alpha,\beta}(t)$ is well defined for all $t \in \mathbb{R}_+$. In order to see that it is, on an event of probability 1, an upper-semi-continuous function, it is enough to prove its upper-semi-continuity on $[0,T]$ for every $T \in (0,\infty)$. Fixing such T, we denote by $U_{(j,T)}^{(\alpha)}$ the jth largest value of $U_j^{(\alpha)}$ for which $V_j^{(\beta)} \in [0,T]$, $j = 1, 2, \ldots$. We write for $m = 1, 2, \ldots$,

$$\begin{aligned} \eta^{\alpha,\beta}(t) &= \sum_{j=1}^m U_{(j,T)}^{(\alpha)} \mathbf{1}_{\left\{t \in \widetilde{R}_j^{(\beta)}\right\}} + \sum_{j=m+1}^\infty U_{(j,T)}^{(\alpha)} \mathbf{1}_{\left\{t \in \widetilde{R}_j^{(\beta)}\right\}} \\ &=: \eta_{1,m}^{\alpha,\beta}(t) + \eta_{2,m}^{\alpha,\beta}(t), \, t \in [0,T]. \end{aligned}$$

The random function $\eta_{1,m}^{\alpha,\beta}$ is, for every m, upper-semi-continuous as a finite sum of upper-semi-continuous functions. Furthermore, on an event of probability 1,

$$\sup_{t \in [0,T]} \left|\eta_{2,m}^{\alpha,\beta}(t)\right| \leq \sum_{j=m+1}^{m+\ell_\beta} U_{(j,T)}^{(\alpha)} \to 0$$

as $m \to \infty$, whence the upper-semi-continuity of $\eta^{\alpha,\beta}$. We define the random sup-measure corresponding to $\eta^{\alpha,\beta}$ by

$$\eta^{\alpha,\beta}(G) := \sup_{t \in G} \eta^{\alpha,\beta}(t), \; G \in \mathcal{G}, \; \text{the collection of open subsets of } \mathbb{R}_+.$$

As usually, one may extend, if necessary, the domain of $\eta^{\alpha,\beta}$ to all subsets of $\mathbb{R}_+$. We emphasize that we use the same notation $\eta^{\alpha,\beta}$ for both the random upper-semi-continuous function and the random sup-measure without causing too much ambiguity, thanks to the homeomorphism between the spaces $\mathrm{SM}(\mathbb{R}_+)$ and $\mathrm{USC}(\mathbb{R}_+)$. It remains to show the measurability of $\eta^{\alpha,\beta}$. Recall that the sup-vague topology of $\mathrm{SM} \equiv \mathrm{SM}(\mathbb{R}_+)$ has sub-bases consisting of

$$\{m \in \mathrm{SM} : m(K) < x\}, \{m \in \mathrm{SM} : m(G) > x\}, K \in \mathcal{K}, G \in \mathcal{G}, x \in \mathbb{R}_+.$$

See for example Vervaat (1997), Section 3. Then, for every $x > 0$,

$$\{\eta^{\alpha,\beta}(K) < x\} = \bigcap_{S\subset\mathbb{N}} \left(\left\{ \sum_{j\in S} U_j^{(\alpha)} < x \right\} \cap \{I_S \cap K \neq \emptyset\} \right)$$

is clearly measurable for $K \in \mathcal{K}$, and so is $\{\eta^{\alpha,\beta}(G) > x\}$ for $G \in \mathcal{G}$. The measurability thus follows.

Proposition 2. *The random sup-measure $\eta^{\alpha,\beta}$ is stationary and H-self-similar with $H = (1-\beta)/\alpha$.*

Proof. To prove the stationarity of $\eta^{\alpha,\beta}$ as a random sup-measure it is enough to prove that the random upper-semi-continuous function $\eta^{\alpha,\beta}$ defined in (8) has a shift-invariant law. Let $r > 0$ and consider the upper-semi-continuous function $(\eta^{\alpha,\beta}(t+r))_{t\in\mathbb{R}_+}$. Note that

$$\eta^{\alpha,\beta}(t+r) = \sum_{j=1}^{\infty} U_j^{(\alpha)} \mathbf{1}_{\left\{t+r\in\widetilde{R}_j^{(\beta)}\right\}} = \sum_{j=1}^{\infty} U_j^{(\alpha)} \mathbf{1}_{\left\{t\in G_r\left(\widetilde{R}_j^{(\beta)}\right)\right\}},\ t \in \mathbb{R}_+,$$

where G_r is a map from $\mathcal{F}(\mathbb{R}_+)$ to $\mathcal{F}(\mathbb{R}_+)$, defined by

$$G_r(F) := F \cap [r,\infty) - r\,.$$

However, by Proposition 4.1 (c) in Lacaux and Samorodnitsky (2016), the map

$$(x, F) \to \big(x, G_r(F)\big)$$

on $\mathbb{R}_+ \times \mathcal{F}(\mathbb{R}_+)$ leaves the mean measure of the Poisson random measure determined by $(U_j^{(\alpha)}, \widetilde{R}_j^{(\beta)})_{j\in\mathbb{N}}$ unaffected. Hence, the law of the random upper-semi-continuous function $\big(\eta^{\alpha,\beta}(t+r)\big)_{t\in\mathbb{R}_+}$ coincides with that of $\big(\eta^{\alpha,\beta}(t)\big)_{t\in\mathbb{R}_+}$.

Similarly, in order to prove the H-self-similarity of $\eta^{\alpha,\beta}$ as a random sup-measure it is enough to prove that the random upper-semi-continuous function $\eta^{\alpha,\beta}$ defined in (8) is H-self-similar. To this end, let $a > 0$, and note that

$$\eta^{\alpha,\beta}(at) = \sum_{j=1}^{\infty} U_j^{(\alpha)} \mathbf{1}_{\left\{at\in\widetilde{R}_j^{(\beta)}\right\}} = \sum_{j=1}^{\infty} U_j^{(\alpha)} \mathbf{1}_{\left\{t\in a^{-1}V_j^{(\beta)} + a^{-1}R_j^{(\beta)}\right\}},\ t \in \mathbb{R}_+.$$

It follows from Proposition 4.1 (b) in Lacaux and Samorodnitsky (2016) that the Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{F}(\mathbb{R}_+)$ with points

$$\Big(U_j^{(\alpha)}, a^{-1}V_j^{(\beta)}, a^{-1}R_j^{(\beta)}\Big)_{j\in\mathbb{N}}$$

has the same mean measure as the Poisson random measure with points

$$\Big(a^{(1-\beta)/\alpha}U_j^{(\alpha)}, V_j^{(\beta)}, R_j^{(\beta)}\Big)_{j\in\mathbb{N}}.$$

Therefore, the random upper-semi-continuous function $\big(\eta^{\alpha,\beta}(at)\big)_{t\in\mathbb{R}_+}$ has a representation

$$\left(a^{(1-\beta)/\alpha} \sum_{j=1}^{\infty} U_j^{(\alpha)} \mathbf{1}_{\left\{t\in\widetilde{R}_j^{(\beta)}\right\}} \right)_{t\in\mathbb{R}_+},$$

and, hence, has the same law as the random upper-semi-continuous function $\big(a^{(1-\beta)/\alpha}\eta^{\alpha,\beta}(t)\big)_{t\in\mathbb{R}_+}$. $\square$

If we restrict the random upper-semi-continuous functions and random measures above to a compact interval, we can use a particularly convenient measurable enumeration of the points of the Poisson process. Suppose, for simplicity, that the compact interval in question is the unit interval $[0,1]$. The Poisson random measure $(U_j^{(\alpha)}, V_j^{(\beta)}, R_j^{(\beta)})_{j\in\mathbb{N}}$ restricted to $\mathbb{R}_+ \times [0,1] \times \mathcal{F}(\mathbb{R}_+)$ can then be viewed as a Poisson point process $(U_j^{(\alpha)})_{j\in\mathbb{N}}$ on $\mathbb{R}_+$ with mean measure $\alpha u^{-(1+\alpha)}\,du$ marked by two independent sequences $(V_j^{(\beta)})_{j\in\mathbb{N}}$ and $(R_j^{(\beta)})_{j\in\mathbb{N}}$ of i.i.d. random variables. The sequence $(R_j^{(\beta)})_{j\in\mathbb{N}}$ is as before, while $(V_j^{(\beta)})_{j\in\mathbb{N}}$ is a sequence of random variables taking values in $[0,1]$ with the common law $\mathbb{P}(V_j^{(\beta)} \le v) = v^{1-\beta}, v \in [0,1]$. Furthermore, we can enumerate the points of so-obtained Poisson random measure according to the decreasing value of the first coordinate, and express $(U_j^{(\alpha)})_{j\in\mathbb{N}}$ as $(\Gamma_j^{-1/\alpha})_{j\in\mathbb{N}}$ with $(\Gamma_j)_{j\in\mathbb{N}}$ denoting the arrival times of the unit rate Poisson process on $(0,\infty)$. This leads to the following representation

$$(\eta^{\alpha,\beta}(t))_{t\in[0,1]} \stackrel{d}{=} \left(\sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \mathbf{1}_{\left\{t\in \widetilde{R}_j^{(\beta)}\right\}}\right)_{t\in[0,1]}. \tag{9}$$

To conclude this section we would like to draw the attention of the reader to the fact that for every fixed $\alpha \in (0,\infty)$, the family of random sup-measures $(\eta^{\alpha,\beta})_{\beta\in(0,1)}$ interpolates certain familiar random sup-measures. On one hand, as $\beta \downarrow 0$, the limit is well known and simple. To see this, notice first that for $(U_j^{(\alpha)}, V_j^{(\beta)}, R_j^{(\beta)})_{j\in\mathbb{N}}$ representing the Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{F}(\mathbb{R}_+)$ with mean measure $\alpha u^{-(1+\alpha)}(1-\beta)v^{-\beta} dv dP_{R^{(\beta)}}$, one can extend the range of parameters to include $\beta = 0$ by setting

$$P_{R^{(0)}} := \delta_{\{0\}}$$

as a probability distribution (unit point mass at $\{0\}$) on $(\mathcal{F}(\mathbb{R}_+), \mathcal{B}(\mathcal{F}(\mathbb{R}_+))$. This is natural as $R^{(\beta)} \Rightarrow \{0\}$ in $\mathcal{F}(\mathbb{R}_+)$ as $\beta \downarrow 0$, which follows, for example, from Kyprianou (2006), Exercise 5.8 (the "zero-stable subordinator" can be thought of as a process staying an exponentially distributed amount of time at zero and then "jumping to infinity".) It then follows that

$$\eta^{\alpha,\beta}(\cdot) \Rightarrow \eta^{\alpha,0}(\cdot) := \bigvee_{j=1}^{\infty} U_j^{(\alpha)} \mathbf{1}_{\left\{V_j^{(0)} \cap \cdot \neq \emptyset\right\}}$$

as $\beta \downarrow 0$.

The random sup-measure $\eta^{\alpha,0}$ above is the independently scattered (a.k.a. completely random) α-Fréchet max-stable random sup-measure on $\mathbb{R}_+$ with Lebesgue measure as the control measure (see Stoev and Taqqu (2005) and Molchanov and Strokorb (2016)). Furthermore, $(\eta^{\alpha,\beta}([0,t]))_{t\ge0}$ corresponds to the extremal process $(Z_\alpha(t))_{t\ge0}$ in (2) for a sequence of i.i.d. random variables with tail index α. The extremal process Z_α also belongs to the class of α-Fréchet max-stable processes (see e.g. de Haan (1984), Kabluchko (2009)).

In the range $\beta \in (0, 1/2]$, the structure of $\eta^{\alpha,\beta}$ can also be simplified. As there are no intersections among independent shifted β-stable regenerative sets, the random sup-measure on the positive real line becomes

$$\eta^{\alpha,\beta}(\cdot) = \bigvee_{j=1}^{\infty} U_j^{(\alpha)} \mathbf{1}_{\left\{\widetilde{R}_j^{(\beta)} \cap \cdot \neq \emptyset\right\}}, \quad \beta \in (0, 1/2].$$

This random sup-measure was first studied in Lacaux and Samorodnitsky (2016). This is an α-Fréchet max-stable random sup-measure, belonging to the class of the so-called Choquet random sup-measures introduced in Molchanov and Strokorb (2016). It is also known that for $\beta \in (0, 1/2]$, $(\eta^{\alpha,\beta}([0, t]))_{t\geq 0}$ has the same distribution as the time-changed extremal process $(Z_\alpha(t^{1-\beta}))_{t\geq 0}$; see Owada and Samorodnitsky (2015b) and Lacaux and Samorodnitsky (2016).

On the other hand, as soon as $\beta > 1/2$, the random sup-measure $\eta^{\alpha,\beta}$ is no longer an α-Fréchet random sup-measure, due to the appearance of intersections. As $\beta \uparrow 1$, the sets $\widetilde{R}^{(\beta)}$ become larger and larger in terms of Hausdorff dimension, and more and more $U_j^{(\alpha)}$s enter the sums defining the random measure due to intersections of more and more $\widetilde{R}_j^{(\beta)}$. In the limit, $\widetilde{R}^{(\beta)} \Rightarrow [0, \infty)$ in $\mathcal{F}(\mathbb{R}_+)$ as $\beta \uparrow 1$ (the "one-stable subordinator" is just the straight line). In the limit, therefore, all $U_j^{(\alpha)}$s contribute to the sum determining the random sup-measure, but for the infinite sum to be finite, restricting ourselves to the case $\alpha \in (0, 1)$ is necessary. In this case we have

$$\eta^{\alpha,\beta}(\cdot) \Rightarrow \eta^{\alpha,1}(\cdot) := \left(\sum_{j=1}^{\infty} U_j^{(\alpha)}\right) \mathbf{1}_{\{\cdot \cap \mathbb{R}_+ \neq \emptyset\}}$$

as $\beta \uparrow 1$. In words, the limit is a random sup-measure with *complete dependence* that takes the same value $\sum_{j=1}^{\infty} U_j^{(\alpha)}$ on every open interval. Note that this random series follows the totally skewed α-stable distribution.

In particular, for every $\alpha \in (0, 1)$, the distributions of random variables $(\eta^{\alpha,\beta}((0, 1)))_{\beta\in[0,1]}$ interpolate between the α-Fréchet distribution ($\beta = 0$) and the totally skewed α-stable distribution ($\beta = 1$). These distributions, to the best of our knowledge, have not been described before. Their properties will be the sbject of future investigations. The tail behaviour of $\eta^{\alpha,\beta}((0, 1))$ is, however, clear, and it is described in the following simple result.

Proposition 3. *For all $\alpha \in (0, \infty)$,*

$$x^\alpha \mathbb{P}\left(\eta^{\alpha,\beta}((0, 1)) > x\right) \to 1$$

as $x \to \infty$.

Proof. Consider the representation (9). Since $\mathbb{P}(\widetilde{R}^{(\beta)} \cap (0, 1) \neq \emptyset) = 1$, with probability one

$$\Gamma_1^{-1/\alpha} \leq \eta^{\alpha,\beta}((0, 1)) \leq \Gamma_1^{-1/\alpha} + (\ell_\beta - 1)\Gamma_2^{-1/\alpha}.$$

Note that $\mathbb{P}(\Gamma_1^{-1/\alpha} > x) \sim x^{-\alpha}$ as $x \to \infty$, and that for $\delta \in (0, \alpha)$,

$$\mathbb{P}\left(\Gamma_2^{-1/\alpha} > x\right) \leq \frac{\mathbb{E}\Gamma_2^{-(\alpha+\delta)/\alpha}}{x^{\alpha+\delta}} = \frac{\Gamma(1 - \delta/\alpha)}{x^{\alpha+\delta}}, \quad x > 0,$$

where $\Gamma(x)$ is the Gamma function. Hence the result. □

As we shall see below, for each α, β the random sup-measure $\eta^{\alpha,\beta}$ arises in the limit of the extremes of stationary processes: while α indicates the tail behavior,

β indicates the length of memory. The limiting case $\beta = 0$ corresponds to the short memory case already extensively investigated in the literature, and the case $\beta \in (0,1)$ corresponds to the long range dependence regime. The larger the β is, the longer the memory becomes.

4. A family of stationary infinitely divisible processes

We consider a discrete-time stationary symmetric infinitely divisible process whose function space Lévy measure is based on an underlying null-recurrent Markov chain. Similar models have been investigated in the symmetric α-stable (SαS) case in Resnick et al. (2000), Samorodnitsky (2004) Owada and Samorodnitsky (2015a,b), Owada (2016) and Lacaux and Samorodnitsky (2016), which can be consulted for various background facts stated below. We first describe the Markov chain. Consider an irreducible aperiodic null-recurrent Markov chain $(Y_n)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}$ with $\mathbb{N}_0 = \{0\}\cup\mathbb{N}$. Fix a state i_0, and let $(\pi_i)_{i\in\mathbb{Z}}$ be the unique invariant measure on $\mathbb{Z}$ such that $\pi_{i_0} = 1$. Consider the space $(E,\mathcal{E}) = (\mathbb{Z}^{\mathbb{N}_0}, \mathcal{B}(\mathbb{Z}^{\mathbb{N}_0}))$. We denote each element of E by $x \equiv (x_0, x_1, \dots)$. Let P_i denote the probability measure on $(E,\mathcal{E})$ determined by the Markov chain starting at $Y_0 = i$, and introduce an infinite σ-finite measure on $(E,\mathcal{E})$ defined by

$$\mu(B) := \sum_{i\in\mathbb{Z}} \pi_i P_i(B),\ B \in \mathcal{E}.$$

Consider

$$A_0 := \{x \in E : x_0 = i_0\},$$

and the first entrance time of A_0

$$\varphi_{A_0}(x) := \inf\{n \in \mathbb{N} : x_n = i_0\},\ x \in E.$$

The key assumption is, for some $\beta \in (0,1)$, $\sum_{k=1}^n P_{i_0}(\varphi_{A_0} \geq k) \in RV_{1-\beta}$, which is equivalent to $P_{i_0}(\varphi_{A_0} \geq k) \in RV_{-\beta}$. Here and in the sequel, $RV_{-\alpha}$ stands for the family of functions on $\mathbb{N}_0$ that are regularly varying at infinity with index $-\alpha$. This assumption can also be expressed in terms of the so-called *wandering rate sequence* defined by

$$w_n := \mu\left(\bigcup_{k=0}^{n-1} \{x \in E : x_k = i_0\}\right), n \in \mathbb{N}.$$

Then

$$w_n \sim \mu(\varphi_{A_0} \leq n) \sim \sum_{k=1}^n P_{i_0}(\varphi_{A_0} \geq k), \tag{10}$$

and the key assumption means that $w_n \in RV_{1-\beta}$. If T denotes the shift operator $T(x_0, x_1, \dots) = (x_1, x_2, \dots)$, then μ is T-invariant: $\mu(\cdot) = \mu(T^{-1}\cdot)$ on $(E,\mathcal{E})$. Furthermore, T is conservative and ergodic with respect to μ on $(E,\mathcal{E})$.

Next, we shall consider non-negative functions from $L^\infty(\mu)$ supported by A_0. Fix $\alpha > 0$. For a fixed $f \in L^\infty(\mu)$, write

$$b_n := \left(\int \max_{k=0,\dots,n} \left(f \circ T^k(x)\right)^\alpha \mu(dx)\right)^{1/\alpha}, n \in \mathbb{N}. \tag{11}$$

The sequence (b_n) satisfies

$$\lim_{n\to\infty} \frac{b_n^\alpha}{w_n} = \|f\|_\infty. \tag{12}$$

Given a Markov chain as above and $f \in L^\infty(\mu)$ supported by A_0, we define a stationary symmetric infinitely divisible process as a stochastic integral

$$X_n := \int_E f_n(x) M(dx) \quad \text{with} \quad f_n := f \circ T^n,\ n \in \mathbb{N}_0, \tag{13}$$

where M is a homogeneous symmetric infinitely divisible random measure on $(E, \mathcal{E})$ with control measure μ and a local symmetric Lévy measure ρ satisfying

$$\rho\big((z,\infty)\big) = az^{-\alpha} \quad \text{for } z \geq z_0 > 0. \tag{14}$$

We refer the reader to Chapter 3 in Samorodnitsky (2016) for more details on integrals with respect to infinitely divisible random measures and, in particular, for the fact that the stochastic process in (13) is a well defined stationary infinitely divisible process. In particular, this process satisfies

$$\mathbb{P}(X_0 > x) \sim a\|f\|_\alpha^\alpha\, x^{-\alpha}$$

as $x \to \infty$; see Rosiński and Samorodnitsky (1993). We will use the value of α defined by (14) in (11). Below we will work with a more explicit and helpful series representation, (16) of the processes of interest.

We would like to draw the attention of the reader to the fact that we are assuming in (14) that the tail of the local Lévy measure has, after a certain point, exact power-law behavior. This is done purely for clarity of the presentation. There is no doubt whatsoever that limiting results similar to the one we prove in the next section hold under a much more general assumption of the regular variation of the tail of ρ. However, the analysis in this case will involve additional layers of approximation that might obscure the nature of the new limiting process we will obtain (note, however, that the assumption (14) already covers the SαS case when $\alpha \in (0,2)$.) In a similar vein, for the sake of clarity, we will assume in the next section that f is simply the indicator function of the set A_0.

Other types of limit theorems for this and related class of processes have been investigated for the partial sums (by Owada and Samorodnitsky (2015a); Jung et al. (2016)) and for the sample covariance functions (by Resnick et al. (2000); Owada (2016)). In all cases non-standard normalizations, or even new limit processes, show up in the limit theorems, indicating long range dependence in the model. Properties of stationary infinitely divisble processes have intrinsic connections to infinite ergodic theory (see Rosiński (1995); Samorodnitsky (2005); Kabluchko and Stoev (2016)), and the family of processes we are considering are said to be *driven by a null-recurrent flow*. The mixing properties of such processes (in the SαS case with $\alpha \in (0,2)$) were investigated in Rosiński and Samorodnitsky (1996).

5. A limit theorem for stationary infinitely divisible processes

Consider the stationary infinitely divisible process introduced in (13). For $n = 1, 2, \ldots$ we define a random sup-measure by

$$M_n(B) := \max_{k \in nB} X_k,\ B \subset [0,\infty)\,.$$

The main result of this paper is the following theorem.

Theorem 4. *Consider the stationary infinitely divisible process $(X_n)_{n\in\mathbb{N}_0}$ defined in the previous section. Let $f = \mathbf{1}_{A_0}$ with $A_0 = \{x \in E : x_0 = i_0\}$. Under the* *assumption that the wandering rate sequence* (w_n) *in* (14) *is regularly varying with index* $1-\beta \in (0,1)$ *and the assumption* (10),

$$\frac{1}{b_n} M_n \Rightarrow a^{1/\alpha} \eta^{\alpha,\beta}$$

as $n \to \infty$ *in the space* $\mathrm{SM}(\mathbb{R}_+)$ *, where* a *is as in* (14).

We start with some preparation. Note that by (12), $b_n^\alpha \in RV_{1-\beta}$. By stationarity it suffices to prove convergence in the space $\mathrm{SM}([0,1])$. We start by decomposing the process $(X_n)_{n\in\mathbb{N}_0}$ into the sum of two independent stationary symmetric infinitely divisible processes:

$$X_n = X_n^{(1)} + X_n^{(2)},\, n \in \mathbb{N}_0,$$

with

$$X_n^{(i)} := \int_E f_n(x) M^{(i)}(dx),\, n \in \mathbb{N}_0,\, i = 1, 2,$$

with f_n as in (13), and $M^{(1)}$ and $M^{(2)}$ two independent homogeneous symmetric infinitely divisible random measures on $(E, \mathcal{E})$, each with control measure μ. The local Lévy measure for $M^{(1)}$ is the measure ρ restricted to the set $\{|z| \ge z_0\}$, while the local Lévy measure for $M^{(2)}$ is the measure ρ restricted to the set $\{|z| < z_0\}$. The first observation is that random variables $(X_n^{(2)})_{n\in\mathbb{N}_0}$ have Lévy measures supported by a bounded set, hence they have exponentially fast decaying tails; see for example Sato (1999). Therefore,

$$\frac{1}{b_n} \max_{k=0,1,\ldots,n} |X_k^{(2)}| \to 0$$

in probability as $n \to \infty$. Therefore, without loss of generality we may assume that, in addition to (14), the local Lévy measure ρ is, to start with, supported by the set $\{|z| \ge z_0\}$.

For each $n \in \mathbb{N}$, the random vector $(X_0, \ldots, X_n)$ admits a series representation that we will now describe. For $x > 0$ let

$$G(x) := \inf\big\{z > 0 :\, \rho\big((z,\infty)\big) \le x\big\}.$$

The assumption that ρ is supported by the set $\{|z| \ge z_0\}$ means that

$$G(x) = \begin{cases} a^{1/\alpha} x^{-1/\alpha} & 0 < x < a z_0^{-\alpha} \\ 0 & x \ge a z_0^{-\alpha}. \end{cases} \tag{15}$$

It follows from Theorem 3.4.3 in Samorodnitsky (2016) that the following representation in law holds:

$$(X_k)_{k=0,\ldots,n} \stackrel{d}{=} \left(\sum_{j=1}^\infty \varepsilon_j G\big(\Gamma_j / 2b_n^\alpha\big) \mathbf{1}_{\left\{T^k(U_j^{(n)})_0 = i_0\right\}} \right)_{k=0,\ldots,n}, \tag{16}$$

where $(\Gamma_j)_{j\in\mathbb{N}}$ are as in (9), $(\varepsilon_j)_{j\in\mathbb{N}}$ are i.i.d. Rademacher random variables and $(U_j^{(n)})_{j\in\mathbb{N}}$ are i.i.d. E-valued random variables with common law μ_n, determined by

$$\frac{d\mu_n}{d\mu}(x) = \frac{1}{b_n^\alpha} \mathbf{1}_{\{T^k(x)_0 = i_0 \ \text{ for some } k = 0,1,\ldots,n\}},\, x \in E.$$

All three sequences are independent. Here and in the sequel, for $x \in E \equiv \mathbb{Z}^{\mathbb{N}_0}$ we write $T^k(x)_0 \equiv [T^k(x)]_0 \in \mathbb{Z}$.

Our argument consists of coupling the series representation of $\eta^{\alpha,\beta}$ in (9) with the series representation of the process in (16). It proceeds through a truncation argument. Introduce for $\ell = 1, 2, \ldots$

$$M_{\ell,n}(B) := \max_{k\in nB} \sum_{j=1}^{\ell} \varepsilon_j G\big(\Gamma_j/2b_n^\alpha\big)\mathbf{1}_{\left\{T^k(U_j^{(n)})_0 = i_0\right\}}, n \in \mathbb{N},$$

and

$$\eta_\ell^{\alpha,\beta}(t) := \sum_{j=1}^{\ell} \Gamma_j^{-1/\alpha}\mathbf{1}_{\left\{t\in \widetilde{R}_j^{(\beta)}\right\}},\ t \in [0,1].$$

We also let $\eta_\ell^{\alpha,\beta}$ denote the corresponding truncated random sup-measure. The key steps of the proof of Theorem 4 are Propositions 5 and 6 below.

Proposition 5. *Under the assumptions of Theorem 4, for all $\ell \in \mathbb{N}$,*

$$\frac{1}{b_n} M_{\ell,n} \Rightarrow a^{1/\alpha}\eta_\ell^{\alpha,\beta}$$

as $n \to \infty$ in the space of $\mathrm{SM}([0,1])$.

Proposition 6. *Under the assumptions of Theorem 4, for all $\delta > 0$,*

$$\lim_{\ell\to\infty}\limsup_{n\to\infty} \mathbb{P}\left(\max_{k=0,\ldots,n} \frac{1}{b_n}\left|\sum_{j=\ell+1}^{\infty} \varepsilon_j G\big(\Gamma_j/2b_n^\alpha\big)\mathbf{1}_{\left\{T^k(U_j^{(n)})_0 = i_0\right\}}\right| > \delta\right) = 0.$$

We start with several preliminary results needed for the proof of Proposition 5. First of all we establish convergence of intersections of trajectories of the Markov chain. Introduce

$$\widehat{R}_{j,n}^{(\beta)} := \frac{1}{n}\left\{k \in \{0,\ldots,n\} : T^k(U_j^{(n)})_0 = i_0\right\},$$

$$\widehat{I}_{S,n} := \bigcap_{j\in S}\widehat{R}_{j,n}^{(\beta)}, S \subset \mathbb{N}, S \neq \emptyset \quad \text{and} \quad \widehat{I}_{\emptyset,n} := \frac{1}{n}\left\{0, 1, \ldots, n\right\}.$$

Recall the definition of I_S in (7).

Lemma 7. *Under the notation above, for all $\ell \in \mathbb{N}$,*

$$\left(\widehat{I}_{S,n}\right)_{S\subset\{1,\ldots,\ell\}} \Rightarrow \left(I_S \cap [0,1]\right)_{S\subset\{1,\ldots,\ell\}} \tag{17}$$

as $n \to \infty$ in $\mathcal{F}([0,1])^{2^\ell}$.

Proof. Recall that $\widetilde{R}_j^{(\beta)} = V_j^{(\beta)} + R_j^{(\beta)}$. We first prove

$$\widehat{R}_{j,n}^{(\beta)} \Rightarrow \widetilde{R}_j^{(\beta)} \cap [0,1], j \in \mathbb{N}. \tag{18}$$

This has been shown in Lacaux and Samorodnitsky (2016), although not in the framework of convergence of in $\mathcal{F}([0,1])$. We sketch the proof below. We can write

$$\widehat{R}_{j,n}^{(\beta)} = \left(V_{j,n}^{(\beta)} + R_{j,n}^{(\beta)}\right) \cap [0,1]$$

with

$$V_{j,n}^{(\beta)} := \frac{1}{n}\min\left\{k = 0,\ldots,n : T^k(U_j^{(n)})_0 = i_0\right\},$$

$$R_{j,n}^{(\beta)} := \frac{1}{n}\left\{k = 0,\ldots,n : T^{nV_{j,n}^{(\beta)}+k}(U_j^{(n)})_0 = i_0\right\}.$$

By the strong Markov property, conditioning on $V_{j,n}^{(\beta)}$, we have

$$R_{j,n}^{(\beta)} \stackrel{d}{=} \frac{1}{n}\left\{k \in \{0,\dots,n\} : T^k(U^{(n)})_0 = i_0\right\},$$

where $U^{(n)}$ is an E-valued random variable with law P_{i_0}, It was shown in Lacaux and Samorodnitsky (2016), Proposition 4.5 and Remark 4.6 that as $n \to \infty$,

$$R_{j,n}^{(\beta)} \Rightarrow R_j^{(\beta)} \cap [0,1]$$

in $\mathcal{F}([0,1])$, and it is clear that $V_{j,n}^{(\beta)} \Rightarrow V_j^{(\beta)}$. Therefore, (18) follows.

Next, we show that for every finite $S \subset \mathbb{N}$

$$\widehat{I}_{S,n} \Rightarrow I_S \cap [0,1]\,. \tag{19}$$

The case $S = \emptyset$ is trivial. It suffices to consider, for all $\ell \in \mathbb{N}$, $S = \{1,\dots,\ell\}$. The claim will follow once we show for all $m \in \mathbb{N}$ and all fixed mutually disjoint open intervals $T_1,\dots,T_m$ in $[0,1]$,

$$\lim_{n\to\infty} \mathbb{P}\left(\bigcap_{i=1}^m \left\{\widehat{I}_{S,n} \cap T_i \neq \emptyset\right\}\right) = \mathbb{P}\left(\bigcap_{i=1}^m \{I_S \cap T_i \neq \emptyset\}\right). \tag{20}$$

To show this, we apply the continuous mapping theorem. Introduce, for $T_1,\dots,T_m$ fixed as above,

$$J(F_1,\dots,F_\ell) := \mathbf{1}_{\left\{\bigcap_{i=1}^m\left\{(\bigcap_{j=1}^\ell F_j)\cap T_i \neq \emptyset\right\}\right\}} = \prod_{i=1}^m \mathbf{1}_{\left\{\bigcap_{j=1}^\ell F_j \cap T_i \neq \emptyset\right\}}, \quad F_j \in \mathcal{F}([0,1]).$$

Each indicator on the right-hand side above is measurable, and hence so is J. Now (20) becomes

$$\lim_{n\to\infty} \mathbb{E}J\left(\widehat{R}_{1,n}^{(\beta)},\dots,\widehat{R}_{\ell,n}^{(\beta)}\right) = \mathbb{E}J\left(\widetilde{R}_1^{(\beta)} \cap [0,1],\dots,\widetilde{R}_\ell^{(\beta)} \cap [0,1]\right). \tag{21}$$

Even though the function J is not continuous on $\mathcal{F}([0,1])^\ell$, we claim that it is continuous a.s. with respect to the law of $\left(\widetilde{R}_\ell^{(\beta)} \cap [0,1]\right)_{j=1,\dots,\ell}$. To see this note that any point $(F_1,\dots,F_\ell)$ in $\mathcal{F}([0,1])^\ell$ such that $F_j \cap T_i \neq \emptyset$ for each $j = 1,\dots,\ell$ and each $i = 1,\dots,m$ is a continuity point of J by the definition of the Fell topology. Furthermore, so is every point $(F_1,\dots,F_\ell)$ in $\mathcal{F}([0,1])^\ell$ with the following property: if $F_j \cap T_i = \emptyset$ for some $j = 1,\dots,\ell$ and $i = 1,\dots,m$, then the set F_j is separated from the interval T_i, i.e. there is $\epsilon > 0$ such that $F_j \cap [a_i - \epsilon, b_i + \epsilon] = \emptyset$ if $T_i = [a_i, b_i]$. This claim follows from Proposition B.3 in Molchanov (2005). If a closed set does not hit a bounded open interval, it is automatically separated from the latter as long as it does not contain any of the endpoints of that interval. Therefore, the only points $(F_1,\dots,F_\ell)$ in $\mathcal{F}([0,1])^\ell$ that are not continuity points of J are those for which at least one of the sets, say F_j, does not hit one of the open intervals, but does contain one of its endpoints. Since a stable subordinator does not hit fixed points, it follows that the function J is continuous a.s. with respect to the law of $\left(\widetilde{R}_\ell^{(\beta)} \cap [0,1]\right)_{j=1,\dots,\ell}$, and the continuous mapping theorem applies. That is, (21) holds and thus (19) follows. Since proving the joint convergence in (17) is only notationally different from the above argument, the proof is complete. $\square$

Next, we show that for each open interval T, outside an event $A_n(T)$ to be defined below, of which the probability tends to zero as $n \to \infty$, the following key identity holds:

(22)
$$\max_{k\in nT}\sum_{j=1}^{\ell}\varepsilon_j G\big(\Gamma_j/2b_n^\alpha\big)\mathbf{1}_{\left\{T^k(U_j^{(n)})_0=i_0\right\}} = \max_{S\subset\{1,\dots,\ell\}}\mathbf{1}_{\left\{\widehat I_{S,n}\cap T\neq\emptyset\right\}}\sum_{j\in S}\varepsilon_j G\big(\Gamma_j/2b_n^\alpha\big)$$
$$= \max_{S\subset\{1,\dots,\ell\}}\mathbf{1}_{\left\{\widehat I_{S,n}\cap T\neq\emptyset\right\}}\sum_{j\in S}\mathbf{1}_{\{\varepsilon_j=1\}}G\big(\Gamma_j/2b_n^\alpha\big),$$

with the convention that $\sum_{j\in\emptyset}=0$.

To establish this, we take a closer look at the simultaneous returns of Markov chain to i_0. We say that the chain indexed by j returns to i_0 at time k, if $T^k(U_j^{(n)})_0 = i_0$. Note that if

$$\frac{k}{n}\in\widehat I_{S,n}\cap T = \left(\bigcap_{j\in S}\widehat R_{j,n}^{(\beta)}\right)\cap T,$$

then there might be another $j'\in\{1,\dots,\ell\}\setminus S$, such that the chain indexed by j' returns to i_0 at the same time k as well. We need an exact description of simultaneous returns of multiple chains. For this purpose, introduce

$$\widehat I_{S,n}^* := \widehat I_{S,n}\cap\left(\bigcup_{j\in\{1,\dots,\ell\}\setminus S}\widehat R_{j,n}^{(\beta)}\right)^c,$$

the collection of all time points (divided by n) at which all chains indexed by S, and only these chains, return to i_0 simultaneously. We define the event

$$A_n(T) := \bigcup_{S\subset\{1,\dots,\ell\}}\left(\left\{\widehat I_{S,n}\cap T\neq\emptyset\right\}\cap\left\{\widehat I_{S,n}^*\cap T=\emptyset\right\}\right). \tag{23}$$

In words, on the complement of $A_n(T)$, if $\widehat I_{S,n}\cap T\neq\emptyset$ for some non-empty set S, then at some time point $k\in nT$, exactly those chains indexed by S return to i_0.

Lemma 8. *For every open interval T, the identity* (22) *holds on $A_n(T)^c$, and* $\lim_{n\to\infty}\mathbb{P}(A_n(T))=0$.

Proof. We first prove the first part of the lemma. Noticing that $S=\emptyset$ is also included in the union above, and

$$\widehat I_{\emptyset,n}^* = \left(\bigcup_{j=1,\dots,\ell}\widehat R_{j,n}^{(\beta)}\right)^c,$$

we see that $A_n(T)$ includes the event that at every time k at least one of the ℓ chains returns to i_0. So on $A_n(T)^c$, the first two terms in (22), which are, clearly, always equal, are non-negative. Furthermore, when $\widehat I_{S,n}\cap T\neq\emptyset$ for some non-empty S, then for $S' := \{j\in S:\varepsilon_j=1\}\subset S$, $\widehat I_{S,n}\cap T\neq\emptyset$ implies $\widehat I_{S',n}\cap T\neq\emptyset$, and therefore restricted to the event $A_n(T)^c$ we have $\widehat I_{S',n}^*\cap T\neq\emptyset$. It follows that the second equality in (22) also holds on $A_n(T)^c$.

For the second part of the lemma, in view of (23), it suffices to show for all S,

$$\lim_{n\to\infty} \mathbb{P}\left(\left\{\widehat{I}_{S,n} \cap T \neq \emptyset\right\} \cap \left\{\widehat{I}^*_{S,n} \cap T = \emptyset\right\}\right) = 0.$$

The case $S = \emptyset$ is trivial. So without loss of generality, assume $S = \{1, \ldots, \ell'\}$ for some $\ell' \in \{1, 2, \ldots, \ell - 1\}$. Introduce

$$K_n := n \min\left(\widehat{I}_{S,n} \cap T\right),$$

the first time in nT that all chains indexed by S return to i_0 simultaneously. Then,

$$\left\{\widehat{I}_{S,n} \cap T \neq \emptyset\right\} \cap \left\{\widehat{I}^*_{S,n} \cap T = \emptyset\right\} \subset \bigcup_{j=\ell'+1}^{\ell} \left\{T^{K_n}(U_j^{(n)})_0 = i_0, \widehat{I}_{S,n} \cap T \neq \emptyset\right\}.$$

The probability of each event in the union on the right hand side is bounded from above by

$$\mathbb{P}\left(T^{K_n}(U_j^{(n)})_0 = i_0 \;\middle|\; \widehat{I}_{S,n} \cap T \neq \emptyset\right) \le \max_{k=0,\ldots,n} \mathbb{P}\left(T^k(U_1^{(n)})_0 = i_0\right) = b_n^{-\alpha}$$

by the i.i.d. assumption on the chains. Since $b_n \to \infty$, the proof is complete. $\square$

Now we are ready to prove the main result.

Proof of Proposition 5. By Theorem 3.2 in O'Brien et al. (1990) and the fact that the stable regenerative sets do not hit points, it suffices to show, for all $m \in \mathbb{N}$ and all disjoint open intervals $T_i = (t_i, t_i') \subset [0, 1], i = 1, \ldots, m$,

$$\left(\frac{1}{b_n} M_{\ell,n}(T_i)\right)_{i=1,\ldots,m} \Rightarrow \left(a^{1/\alpha} \eta_\ell^{\alpha,\beta}(T_i)\right)_{i=1,\ldots,m} \tag{24}$$

as random vectors in $\mathbb{R}^m$. The expression (15) and the fact that $b_n \to \infty$ tell us that the event $B_n := \{\Gamma_\ell / 2b_n^\alpha < a z_0^{-\alpha}\}$ has probability going to 1 as $n \to \infty$, and on B_n we have $G(\Gamma_j / 2b_n^\alpha) = (2a)^{1/\alpha} \Gamma_j^{-1/\alpha} b_n$. That is,

$$\frac{1}{b_n} M_{\ell,n}(T_i) = \max_{k \in nT_i} \sum_{j=1}^{\ell} \varepsilon_j (2a)^{1/\alpha} \Gamma_j^{-1/\alpha} \mathbf{1}_{\left\{T^k(U_j^{(n)})_0 = i_0\right\}}.$$

Therefore, proving (24) is the same as proving that

$$\left(\max_{k \in nT_i} \sum_{j=1}^{\ell} \varepsilon_j 2^{1/\alpha} \Gamma_j^{-1/\alpha} \mathbf{1}_{\left\{T^k(U_j^{(n)})_0 = i_0\right\}}\right)_{i=1,\ldots,m} \Rightarrow \left(\eta_\ell^{\alpha,\beta}(T_i)\right)_{i=1,\ldots,m}. \tag{25}$$

The first part of Lemma 8 yields that on $A_n(T_i)^c \cap B_n$,

$$\max_{k \in nT_i} \sum_{j=1}^{\ell} \varepsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}_{\left\{T^k(U_j^{(n)})_0 = i_0\right\}} = \max_{S \subset \{1,\ldots,\ell\}} \mathbf{1}_{\left\{\widehat{I}_{S,n} \cap T_i \neq \emptyset\right\}} \sum_{j \in S} \mathbf{1}_{\{\varepsilon_j = 1\}} \Gamma_j^{-1/\alpha}.$$

Note that the point process $\left(\Gamma_j^{-1/\alpha}\right)_{j \in \mathbb{N}, \varepsilon_j = 1}$ is a Poisson point process with mean measure $2^{-1} \alpha u^{-(1+\alpha)}\, du,\ u > 0$, and it can be represented in law as the point process $\left(2^{-1/\alpha}\Gamma_j^{-1/\alpha}\right)_{j\in\mathbb{N}}$. Since by Lemma 8, $\mathbb{P}(A_n(T_i)^c \cap B_n) \to 1$ as $n\to\infty$, the statement (25) will follow once we prove that

$$\left(\max_{S\subset\{1,\dots,\ell\}} \mathbf{1}_{\left\{\widehat{I}_{S,n}\cap T_i\neq\emptyset\right\}} \sum_{j\in S}\Gamma_j^{-1/\alpha}\right)_{i=1,\dots,m} \Rightarrow \left(\eta_\ell^{\alpha,\beta}(T_i)\right)_{i=1,\dots,m}.$$

This is, however, an immediate consequence of Lemma 7 and the fact that $\eta_\ell^{\alpha,\beta}(T_i)$ can be written in the form

$$\eta_\ell^{\alpha,\beta}(T_i) = \max_{S\subset\{1,\dots,\ell\}} \mathbf{1}_{\{I_S\cap T_i\neq\emptyset\}} \sum_{j\in S}\Gamma_j^{-1/\alpha},\ i=1,\dots,m.$$

□

Proof of Proposition 6. For $M>0$ let $D_\ell^M := \{\Gamma_{\ell+1}\geq M\}$. It is clear that $\lim_{\ell\to\infty}\mathbb{P}(D_\ell^M)=1$. We have

$$\mathbb{P}\left(\left\{\max_{k=0,\dots,n}\frac{1}{b_n}\left|\sum_{j=\ell+1}^{\infty}\varepsilon_j G\big(\Gamma_j/2b_n^\alpha\big)\mathbf{1}_{\left\{T^k(U_j^{(n)})_0=i_0\right\}}\right|>\delta\right\}\cap D_\ell^M\right)$$
$$\leq \sum_{k=0}^{n}\mathbb{P}\left(\left\{\left|\sum_{j=\ell+1}^{\infty}\varepsilon_j\Gamma_j^{-1/\alpha}\mathbf{1}_{\left\{\Gamma_j\leq 2ab_n^\alpha z_0^{-\alpha}\right\}}\mathbf{1}_{\left\{T^k(U_j^{(n)})_0=i_0\right\}}\right|>\frac{\delta}{(2a)^{1/\alpha}}\right\}\cap D_\ell^M\right).$$

Note that on the right hand side above, the summand takes the same value for all $k=0,1,\dots,n$. Write $\delta' := \delta/(2a)^{1/\alpha}$. We shall show that, for all $\delta'>0$, one can choose M depending on α, β and δ' only, such that for all ℓ,

$$\limsup_{n\to\infty} n\mathbb{P}\left(\left\{\left|\sum_{j=\ell+1}^{\infty}\varepsilon_j\Gamma_j^{-1/\alpha}\mathbf{1}_{\left\{\Gamma_j\leq 2ab_n^\alpha z_0^{-\alpha}\right\}}\mathbf{1}_{\left\{T^k(U_j^{(n)})_0=i_0\right\}}\right|>\delta'\right\}\cap D_\ell^M\right)=0.$$

The desired result then follows. To show the above, first observe that the probability of interest is bounded from above by

$$\mathbb{P}\left(\sum_{j=1}^{\infty}\Gamma_j^{-1/\alpha}\mathbf{1}_{\left\{M\leq\Gamma_j\leq 2ab_n^\alpha z_0^{-\alpha}\right\}}\mathbf{1}_{\left\{T^k(U_j^{(n)})_0=i_0\right\}}>\delta'\right). \tag{26}$$

Observe that the restriction to $(0,\infty)$ of the point process with the points

$$\left(b_n\Gamma_j^{-1/\alpha}\mathbf{1}_{\left\{T^k(U_j^{(n)})_0=i_0\right\}}\right)_{j\in\mathbb{N}}$$

represents a Poisson random measure on $(0,\infty)$ with intensity $\mu(A_0)\alpha u^{-(\alpha+1)}\,du$, $u>0$, and another representation of the same Poisson random measure is

$$\left(\mu(A_0)^{1/\alpha}\Gamma_j^{-1/\alpha}\right)_{j\in\mathbb{N}}.$$

By definition of the Markov chain, $\mu(A_0) = 1$. Therefore, (26) becomes

$$\mathbb{P}\left(b_n^{-1}\sum_{j=1}^\infty \Gamma_j^{-1/\alpha}\mathbf{1}_{\left\{M/b_n^\alpha\le\Gamma_j\le 2az_0^{-\alpha}\right\}} > \delta'\right) \le \mathbb{P}\left(b_n^{-1}\sum_{j=j_M+1}^\infty \Gamma_j^{-1/\alpha}\mathbf{1}_{\left\{\Gamma_j\le 2az_0^{-\alpha}\right\}} > \delta'/2\right) \tag{27}$$

by taking $j_M := \lfloor M^{1/\alpha}\delta'/2\rfloor$, so that $b_n^{-1}\sum_{j=1}^{j_M}\Gamma_j^{-1/\alpha}\mathbf{1}_{\left\{M/b_n^\alpha\le\Gamma_j\le 2az_0^{-\alpha}\right\}} \le \delta'/2$ with probability one. By Markov inequality, we can further bound (27) by, up to a multiplicative constant depending on δ',

$$b_n^{-p}\mathbb{E}\left(\sum_{j=j_M+1}^\infty \Gamma_j^{-1/\alpha}\mathbf{1}_{\left\{\Gamma_j\le 2az_0^{-\alpha}\right\}}\right)^p .$$

If we choose

$$p > \frac{1}{1-\beta},$$

then $b_n^{-p} = o(n^{-1})$. Since choosing M and, hence, j_M large enough, we can ensure finiteness of the above expectation, this completes the proof. □

Proof of Theorem 4. As in the proof of Proposition 5, it suffices to show, for all $m\in\mathbb{N}$ and all disjoint open intervals $T_i = (t_i, t_i') \subset [0,1], i = 1,\dots,m$,

$$\left(\max_{k\in nT_i}\frac{1}{b_n}\sum_{j=1}^\infty \varepsilon_j G\big(\Gamma_j/2b_n^\alpha\big)\mathbf{1}_{\left\{T^k(U_j^{(n)})_0=i_0\right\}}\right)_{i=1,\dots,m} \Rightarrow \left(a^{1/\alpha}\eta^{\alpha,\beta}(T_i)\right)_{i=1,\dots,m} .$$

We will use Theorem 3.2 in Billingsley (1999). By Proposition 5 and the obvious fact that

$$\left((\eta_\ell^{\alpha,\beta}(T_i)\right)_{i=1,\dots,m} \to \left((\eta^{\alpha,\beta}(T_i)\right)_{i=1,\dots,m}$$

a.s. as $\ell\to\infty$, it only remains to check that for any $i = 1,\dots,m$,

$$\lim_{\ell\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\frac{1}{b_n}\left|\max_{k\in nT_i}\sum_{j=1}^\infty \varepsilon_j G\big(\Gamma_j/2b_n^\alpha\big)\mathbf{1}_{\left\{T^k(U_j^{(n)})_0=i_0\right\}} - \max_{k\in nT_i}\sum_{j=1}^\ell \varepsilon_j G\big(\Gamma_j/2b_n^\alpha\big)\mathbf{1}_{\left\{T^k(U_j^{(n)})_0=i_0\right\}}\right| > \varepsilon\right) = 0 \tag{28}$$

for any $\varepsilon > 0$. However, the above probability dos not exceed

$$\mathbb{P}\left(\frac{1}{b_n}\left|\max_{k\in nT_i}\sum_{j=\ell+1}^\infty \varepsilon_j G\big(\Gamma_j/2b_n^\alpha\big)\mathbf{1}_{\left\{T^k(U_j^{(n)})_0=i_0\right\}}\right| > \varepsilon\right)$$
$$\le\mathbb{P}\left(\frac{1}{b_n}\max_{k=0,\dots,n}\left|\sum_{j=\ell+1}^\infty \varepsilon_j G\big(\Gamma_j/2b_n^\alpha\big)\mathbf{1}_{\left\{T^k(U_j^{(n)})_0=i_0\right\}}\right| > \varepsilon\right) ,$$

and (28) follows from Proposition 6. □

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