Indefinite Summation and the Kronecker Delta

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Abstract

Indefinite summation, together with a generalized version of the Kronecker delta, provide a calculus for reasoning about various polynomial functions that arise in combinatorics, such as the Tutte, chromatic, flow, and reliability polynomials. In this paper we develop the algebraic properties of the indefinite summation operator and the generalized Kronecker delta from an axiomatic viewpoint. Our main result is that the axioms are equationally complete; that is, all equations that hold under the intended interpretations are derivable in the calculus.

1 Introduction

The indefinite summation operator is very much like a discrete version of an indefinite integral. One can sum an algebraic expression over the values of an unspecified finite set of unspecified size X, and the result is a new expression involving the indeterminate X. The summand may contain symbols δ_A, for A a set of variables, a modest generalization of the usual two-variable Kronecker delta δ_{xy}, which constrain the elements of A all to have the same value.

The interaction of the summation operator $\sum_x$ with the δ_A can be described with various algebraic identities. For example,

$$\sum_x (\delta_{xyz} + \delta_{uv}\delta_x) \quad = \quad \delta_{yz} + \delta_{uv}X.$$

Intuitively, $\sum_x \delta_{xyz} = \delta_{yz}$, because if y and z should have the same value, then x will take on that value exactly once in the summation, so the constraint that x, y, z have the same value reduces to the constraint that y and z have the same value; and the singleton constraint δ_x is no constraint at all, so $\delta_x = 1$, therefore $\sum_x \delta_{uv}\delta_x = \sum_x \delta_{uv} = \delta_{uv}X$, where X is the symbolic representation of the size of the set of all possible values for x.

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Indefinite summation, together with the generalized Kronecker delta, provide a calculus for reasoning about various polynomial functions that arise in graph theory, notably the Tutte polynomial and its variants. These polynomials give a succinct encoding of a large amount of interesting information about the graph, such as the number of spanning forests, the number of spanning edge-induced subgraphs, and coloring information. They have a host of applications in computer science, mathematics, and physics [15, 10, 3].

Not surprisingly, Tutte polynomials are highly intractable to compute in general. Jaeger et al. [7] show that evaluating the Tutte polynomial at a given point (x, y) is $\#P$-hard except at points on the curve $(x-1)(y-1) = 1$ and 8 other special points. Evaluation is tractable in graphs of bounded treewidth [2, 12, 1, 11, 10] but with very large constants. Many of these algorithms are combinatorial in nature and rely on graph manipulations (e.g. [1]) or reductions to monadic second-order logic (e.g. [10]).

It is well known that the Tutte polynomial and its variants can be derived using summation formulas involving the Kronecker delta δ_{xy}. The first use of the Kronecker delta in this context seems to be the 1932 paper of Whitney [17], although it was not called by that name. The theory is strongly allied to the theory of Möbius inversion and incidence algebra [13, 14, 4], and to some extent to the calculus of finite differences [8]. In fact it can be cast as a theory of certain linear operators on the Möbius algebra of a semilattice of partitions [6].

Our ultimate goal is to exploit this theory to derive more abstract symbolic algorithms for the Tutte polynomial and its variants in special cases, with the hope that such algorithms might be more efficient or easier to program or analyze. This paper constitutes an initial step in this direction. We study a system of polynomials with indefinite summation operators and the generalized Kronecker delta. We develop the algebraic properties of these constructs from an axiomatic viewpoint. Our main result is a complete axiomatization of their equational theory; that is, all equations that hold under the intended interpretations are derivable in the calculus.

This paper is organized as follows. In Section 2, we develop the algebraic theory of the generalized Kronecker delta and prove several basic results. In Section 3, we introduce an axiomatization and prove that it is sound and complete with respect to the class of intended interpretations. In Section 4, we introduce the indefinite summation operator, extend the axiomatization and the semantics to include summation, and reprove soundness and completeness. Finally, in Section 5, we illustrate the use of the calculus by giving a symbolic derivation of the closed form of the chromatic polynomial.

Our development relies on some elementary concepts from algebra, for which we refer the reader to [9, 16].

2 The Kronecker Delta

Let V be a finite set of size n. Let $A, B, \ldots$ denote subsets of V. The powerset of V is the set of all subsets of V and is denoted 2^V. Let C be another finite set of size m, and let C^V denote the set of functions $f : V \to C$. Intuitively, we regard C as a set of colors and a function $f : V \to C$ as a coloring of the elements of V. The set C^V is the set of all possible colorings.

For $A \subseteq V$, the *Kronecker delta* δ_A is the function $\delta_A : C^V \to \{0, 1\}$ defined by:

$$\begin{aligned} \delta_A(f) &\stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } \forall x, y \in A\ f(x) = f(y) \\ 0, & \text{otherwise.} \end{cases} \\ &= \begin{cases} 1, & \text{if } A \text{ is monochromatic under the coloring } f \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{2.1}$$

This is a modest generalization of the usual Kronecker delta, which is traditionally defined only for two-element sets:

$$\delta_{xy}(f) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } f(x) = f(y) \\ 0, & \text{otherwise.} \end{cases}$$

Here we have written δ_{xy} as an abbreviation for $\delta_{\{x,y\}}$. We will similarly write δ_x for $\delta_{\{x\}}$, δ_{xyz} for $\delta_{\{x,y,z\}}$, etc.

2.1 Axiomatization

The space of functions $F : C^V \to \{0, 1\}$ forms a commutative monoid under the pointwise operations, and the functions δ_A are elements of this monoid. Multiplication on the δ_A satisfies all the laws of commutative monoids (associativity, commutativity, and the identity laws), as well as the following additional properties.

(2.2) $\delta_A \delta_B = \delta_{A \cup B}$ if $A \cap B \neq \varnothing$

(2.3) $\delta_A = 1$ if A is either $\varnothing$ or a singleton.

Equivalently,

(2.4) $\delta_A = \prod_{\substack{B \subseteq A \\ |B|=2}} \delta_B$

(2.5) $\delta_{xy}\delta_{yz} = \delta_{xy}\delta_{yz}\delta_{xz}$ (transitivity).

Lemma 2.1 *Modulo the laws of commutative monoids,* (2.2) *and* (2.3) *are equivalent to* (2.4) *and* (2.5).

Proof. We apply the laws of commutative monoids (associativity, commutativity, identity laws) without comment. Suppose (2.2) and (2.3) hold. Property (2.5) follows immediately from three applications of (2.2):

$$\delta_{xy}\delta_{yz} \quad = \quad \delta_{xyz} \quad = \quad \delta_{xyz}\delta_{xz} \quad = \quad \delta_{xy}\delta_{yz}\delta_{xz}.$$

For (2.4), let $A = \{x_1, \ldots, x_n\}$, $n \geq 0$. If $n = 0$ or $n = 1$, then $\delta_A = 1$ by (2.3), and the right-hand side of (2.4) is also 1 since it is a vacuous product, as there are no subsets of A of size 2. If $n \geq 2$, we can express the right-hand side of (2.4) as the product $\prod_{i<j} \delta_{x_i x_j}$, then use (2.5) (which we have just proved) to eliminate factors $\delta_{x_i x_j}$ in order of decreasing $j - i$ until obtaining a product

$$\delta_{x_1 x_2} \delta_{x_2 x_3} \cdots \delta_{x_{n-1} x_n}.$$

We then use (2.2) inductively to combine adjacent factors until we obtain δ_A.

Conversely, suppose (2.4) and (2.5) hold. Axiom (2.3) follows from (2.4), since as noted above, the right-hand side of (2.4) is 1 when $|A| \leq 1$. For (2.2), suppose $A \cap B \neq \varnothing$. Picking $z = y$ in (2.5) and using (2.3) (which we have just proved), we have

$$\delta_{xy} \quad = \quad \delta_{xy}\delta_{yy} \quad = \quad \delta_{xy}\delta_{yy}\delta_{xy} \quad = \quad \delta_{xy}^2. \tag{2.6}$$

Then

$$\delta_{A \cup B} \quad = \quad \prod_{\substack{x \in A \\ y \in A}} \delta_{xy} \prod_{\substack{x \in A \\ y \in B}} \delta_{xy} \prod_{\substack{x \in B \\ y \in B}} \delta_{xy} \quad = \quad \prod_{\substack{x \in A \\ y \in A}} \delta_{xy} \prod_{\substack{x \in B \\ y \in B}} \delta_{xy} \quad = \quad \delta_A \delta_B.$$

The first equation is just (2.4), using (2.6) to obtain the duplicated factors for x or $y \in A \cap B$. The second follows from several applications of (2.5) to get rid of the middle product, using the fact that $A \cap B \neq \varnothing$. The last is just two applications of (2.4). $\square$

A useful consequence is that if $A \subseteq B$, then $\delta_B = \delta_A \delta_B$. This is immediate from (2.3) if $A = \varnothing$ and (2.2) if $A \neq \varnothing$. In particular, multiplication is *idempotent*: $\delta_A = \delta_A^2$ for any A. Multiplication in the monoid $C^V \to \{0, 1\}$ is idempotent, but it is important to note that we did not use this fact in the proof of Lemma 2.1, but derived the idempotence of the δ_A axiomatically from (2.2)–(2.3) and (2.4)–(2.5).

2.2 Partitions

A *partition* of V is a family of pairwise disjoint subsets of V whose union is V. Partitions are denoted $\pi, \rho, \ldots$. We say that π *refines* ρ and write $\pi \sqsubseteq \rho$ if each set of π is a subset of a set of ρ; equivalently, if each set of ρ is a union of sets of π. Refinement is a partial order, and every pair of partitions has a least upper bound with respect to refinement, denoted $\pi \sqcup \rho$. This is the finest partition refined by both π and ρ. The $\sqsubseteq$-least element is the identity partition $\iota = \{\{x\} \mid x \in V\}$ and the $\sqsubseteq$-greatest element is $\{V\}$. Thus the family of partitions of V forms an upper semilattice $\Pi(V)$. It also forms a lattice, the meet operation being coarsest common refinement, but this is not relevant for our purposes.

For any family $F \subseteq 2^V$, define

$$\delta_F \quad \stackrel{\text{def}}{=} \quad \prod_{A \in F} \delta_A.$$

Theorem 2.2 *The free monoid generated by* $\{\delta_A \mid A \subseteq V\}$*, modulo* (2.2) *and* (2.3) (*or equivalently by Lemma 2.1,* (2.4) *and* (2.5))*, is isomorphic to* $\Pi(V)$*, the upper semilattice of partitions of* V*.*

Proof. It follows from the laws of commutative monoids and the axioms (2.2)–(2.3) or (2.4)–(2.5) that every $F \subseteq 2^V$ generates a unique partition π of V such that $\delta_F = \delta_\pi$. We can form π by starting with $(F - \varnothing) \cup \{\{x\} \mid x \notin \bigcup F\}$, as justified by (2.2), and replacing A, B such that $A \cap B \neq \varnothing$ with $A \cup B$ until no more such steps are possible, as justified by (2.3).

It also follows that $\delta_\pi \delta_\rho = \delta_{\pi \sqcup \rho}$ and $\pi \sqsubseteq \rho$ iff $\delta_\pi \delta_\rho = \delta_\rho$. Also, $\delta_\iota = 1$, where ι is the identity partition. Thus the map that takes π to δ_π constitutes an isomorphism of monoids. □

2.3 Normal Form

Let R be a commutative ring. The space of functions $F : C^V \to R$ also forms a commutative ring under the pointwise operations, and the monoid of functions $C^V \to \{0, 1\}$ embeds faithfully into this ring.

One can also form the polynomial ring $R[\delta_A \mid A \subseteq V]$ and its quotient modulo (2.2)–(2.3) or (2.4)–(2.5). In light of Theorem 2.2, this quotient structure is isomorphic to the semigroup algebra over R of the upper semilattice of partitions of V. This structure is known as the *Möbius algebra* of the semilattice [6]. We will explore this correspondence more fully in Section 3.7.

Lemma 2.3 (Normal Form) *Any polynomial expression in* $R[\delta_A \mid A \subseteq V]$ *is equivalent modulo* (2.2)–(2.3) *or* (2.4)–(2.5) *to an expression of the form* $\sum_\pi a_\pi \delta_\pi$*, where* $a_\pi \in R$*.*

Proof. Multiply out the given expression to obtain a sum of terms of the form $a_F \delta_F$, replace δ_F by δ_π where π is the partition generated by F, and combine like terms. □

3 Soundness and Completeness

Let $R[\delta_A \mid A \subseteq V]$ denote the ring of polynomials with indeterminates δ_A, $A \subseteq V$, and coefficients in R. In this section we show that the axioms (2.2) and (2.3) (equivalently by Lemma 2.1, (2.4) and (2.5)) are equationally sound and complete for all interpretations of the δ_A as functions of the form (2.1). That is, any equation between polynomials in $R[\delta_A \mid A \subseteq V]$ that holds under all such interpretations is provable by equational logic from the axioms. We will also show that the axioms are complete for interpretations over a fixed color class C, provided $|C| \geq n$. Color classes of size less than n require an extra axiom.

Technically, we must include the equational theory of R in our axiomatization. In all our applications, R will be either $\mathbb{Z}$ or a polynomial ring $\mathbb{Z}[X]$ or $\mathbb{Q}[X]$. The equational theory of $\mathbb{Z}$ or $\mathbb{Z}[X]$ consists of nothing beyond the axioms of commutative rings, since these are free commutative rings. For

$\mathbb{Q}[X]$, we must also include arithmetic in $\mathbb{Q}$. The equational theory of R and the axioms of commutative rings will be part of all our axiomatizations, so we will take this for granted and not mention it further.

3.1 Notational Preliminaries

To formulate the theorem precisely, we must distinguish between the formal symbol δ_A and the function $C^V \to \{0,1\}$ it represents. We will henceforth use the notation $[\![\delta_A]\!]_C$ for the latter.

We will also use the notation $\Theta(\varphi)$ for the characteristic function of a proposition φ, defined by

$$\Theta(\varphi) \quad \stackrel{\text{def}}{=} \quad \begin{cases} 1, & \text{if } \varphi \\ 0, & \text{otherwise.} \end{cases}$$

For a coloring $f : V \to C$, let ρ_f be the partition on V induced by f^{-1}:

$$\rho_f \quad \stackrel{\text{def}}{=} \quad \{f^{-1}(c) \mid c \in C,\ f^{-1}(c) \neq \varnothing\}.$$

The elements of ρ_f are the maximal nonempty subsets of V that are monochromatic under the coloring f.

Similarly, for a set $A \subseteq V$, let ρ_A be the partition generated by A, namely $\{A\} \cup \{\{x\} \mid x \notin A\}$, or just ι if $A = \varnothing$.

With this notation, we can rewrite (2.1) as

$$\begin{aligned} [\![\delta_A]\!]_C(f) \quad &\stackrel{\text{def}}{=} \quad \begin{cases} 1, & \text{if } \forall x, y \in A\ f(x) = f(y) \\ 0, & \text{otherwise} \end{cases} \\ &= \quad \Theta(\forall x, y \in A\ f(x) = f(y)) \\ &= \quad \Theta(\rho_A \sqsubseteq \rho_f). \end{aligned} \tag{3.1}$$

Equation (3.1) defines a set map

$$[\![\ ]\!]_C : \{\delta_A \mid A \subseteq V\} \quad \to \quad (C^V \to R),$$

which extends uniquely to a ring homomorphism

$$[\![\ ]\!]_C : R[\delta_A \mid A \subseteq V] \quad \to \quad (C^V \to R). \tag{3.2}$$

This is just the evaluation homomorphism that evaluates a given polynomial p by substituting the values $[\![\delta_A]\!]_C$ for the indeterminates δ_A. Under this extension,

$$[\![\delta_\pi]\!]_C(f) \quad = \quad \Theta(\pi \sqsubseteq \rho_f). \tag{3.3}$$

It is clear from (3.1) and (3.3) that the interpretation $[\![p]\!]_C : C^V \to R$ does not depend on the actual colorings $f : V \to C$, but only on the partitions ρ_f they generate.

3.2 Problem Formulation

Let I be the ideal in $R[\delta_A \mid A \subseteq V]$ generated by the polynomials

$$\delta_A\delta_B - \delta_{A\cup B} \quad \text{for } A, B \subseteq V,\ A \cap B \neq \varnothing, \tag{3.4}$$

$$\delta_x - 1, \quad \text{for all } x \in V, \tag{3.5}$$

$$\delta_\varnothing - 1 \tag{3.6}$$

corresponding to the axioms (2.2) and (2.3). Equivalently, by Lemma 2.1, I is generated by

$$\delta_{xy}\delta_{yz} - \delta_{xy}\delta_{yz}\delta_{xz} \quad \text{for } x, y, z \in V, \tag{3.7}$$

$$\delta_A - \prod_{\substack{B \subseteq A \\ |B|=2}} \delta_B \quad \text{for } A \subseteq V \tag{3.8}$$

corresponding to the axioms (2.4) and (2.5). Let I_m be the ideal generated by I and the polynomial

$$\prod_{|\pi|=m} (1 - \delta_\pi). \tag{3.9}$$

Note that $I_m = I$ for $m = n$, since $\prod_{|\pi|=n}(1 - \delta_\pi) = 1 - \delta_\iota \in I$.

By general considerations of equational logic, a pair of polynomials $p, q \in R[\delta_A \mid A \subseteq V]$ are provably equal from (2.2)–(2.3) (or (2.4)–(2.5)) iff $p - q \in I$; equivalently, if p and q are equated in the quotient ring $R[\delta_A \mid A \subseteq V]/I$.

Semantically, the ring homomorphism $[\![\ ]\!]_C$ equates p and q if their difference is in the kernel

$$\ker\ [\![\ ]\!]_C \quad = \quad \{r \in R[\delta_A \mid A \subseteq V] \mid [\![r]\!]_C = 0\}. \tag{3.10}$$

Thus the inclusion $I \subseteq \bigcap_C \ker\ [\![\ ]\!]_C$ asserts the soundness of the axiomatization over all interpretations $[\![\ ]\!]_C$, and completeness says that $\bigcap_C \ker\ [\![\ ]\!]_C \subseteq I$.

We will show below (Theorems 3.1(i) and 3.4) that the axiomatization is sound and complete in this sense. The proof will also establish that I is complete for any fixed color class C, provided $|C| \geq n$; that is, $I = \ker\ [\![\ ]\!]_C$.

What if $|C| = m < n$? Call C *deficient* in that case. Here the proof of Theorem 3.4 does not go through, and in fact the conclusion of the theorem is no longer true. To obtain a complete axiomatization for deficient C, we must include the extra axiom (3.9). Intuitively, (3.9) says that some partition with m partition elements refines the partition determined by a coloring of V with m or fewer colors. We will show below (Theorems 3.1(ii) and 3.6) that this axiomatization is sound and complete in the sense that $I_m = \ker\ [\![\ ]\!]_C$ for $|C| = m \leq n$.

3.3 Soundness

The first theorem establishes soundness in all cases.

Theorem 3.1 (Soundness I)

(i) *For any* C*,* $I \subseteq \ker [\![\ ]\!]_C$*.*

(ii) *For* C *such that* $|C| \leq m \leq n$*,* $I_m \subseteq \ker [\![\ ]\!]_C$*.*

Proof. (i) It is enough to show that the generators of I are in $\ker [\![\ ]\!]_C$. Let $f : C^V \to R$ and let $A, B \subseteq V$ with $A \cap B \neq \varnothing$.

$$\begin{aligned} [\![\delta_A\delta_B - \delta_{A\cup B}]\!]_C(f) &= \Theta(\rho_A \sqsubseteq \rho_f) \cdot \Theta(\rho_B \sqsubseteq \rho_f) - \Theta(\rho_{A\cup B} \sqsubseteq \rho_f) = 0, \\ [\![1 - \delta_x]\!]_C(f) &= 1 - \Theta(\rho_x \sqsubseteq \rho_f) = 0, \\ [\![1 - \delta_\varnothing]\!]_C(f) &= 1 - \Theta(\rho_\varnothing \sqsubseteq \rho_f) = 0. \end{aligned}$$

(ii) Assume $|C| \leq m \leq n$. We need only show that the remaining generator (3.9) of I_m is in $\ker [\![\ ]\!]_C$.

$$\begin{aligned} [\![\textstyle\prod_{|\pi|=m}(1 - \delta_\pi)]\!]_C(f) &= \prod_{|\pi|=m} (1 - [\![\delta_\pi]\!]_C(f)) \\ &= \prod_{|\pi|=m} (1 - \Theta(\pi \sqsubseteq \rho_f)) \\ &= \Theta(\forall\pi\ |\pi| = m \Rightarrow \pi \not\sqsubseteq \rho_f)) \\ &= \Theta(|\rho_f| > m), \end{aligned} \tag{3.11}$$

which is 0 since $|\rho_f| \leq |C| \leq m$. □

3.4 Completeness

Now we turn to completeness. First we show that the axiomatization represented by I is complete for all interpretations over all color classes or over a single color class with $|C| \geq n$.

Define inductively

$$\varepsilon_\pi \stackrel{\text{def}}{=} \delta_\pi - \sum_{\substack{\pi\sqsubseteq\rho \\ \pi\neq\rho}} \varepsilon_\rho. \tag{3.12}$$

Then

$$\delta_\pi = \sum_{\pi\sqsubseteq\rho} \varepsilon_\rho. \tag{3.13}$$

The expression (3.12) for ε_π can be expanded inductively to give a linear combination of the δ_ρ, $\pi \sqsubseteq \rho$. The coefficient of δ_ρ in this expression can be computed explicitly by Möbius inversion [13]; it is $(-1)^{|\pi|-|\rho|} \prod_{i=1}^{|\pi|} (i-1)!^{n_i}$, where n_i is the number of elements of ρ that contain exactly i elements of π (although we do not need to know this for our development). The following lemma can also be derived from that theory, but it is just as easy to give an explicit proof.

Lemma 3.2 *For all $f : V \to C$, $[\![\varepsilon_\pi]\!]_C(f) = \Theta(\pi = \rho_f)$.*

Proof. By downward induction on $\sqsubseteq$. Assume that the lemma is true for all ρ such that $\pi \sqsubseteq \rho$, $\pi \neq \rho$. Then

$$\begin{aligned}
[\![\varepsilon_\pi]\!]_C(f) &= [\![\delta_\pi - \sum_{\substack{\pi\sqsubseteq\rho\\ \pi\neq\rho}} \varepsilon_\rho]\!]_C(f) \\
&= [\![\delta_\pi]\!]_C(f) - \sum_{\substack{\pi\sqsubseteq\rho\\ \pi\neq\rho}} [\![\varepsilon_\rho]\!]_C(f) \\
&= \Theta(\pi \sqsubseteq \rho_f) - \sum_{\substack{\pi\sqsubseteq\rho\\ \pi\neq\rho}} \Theta(\rho = \rho_f) \\
&= \Theta(\pi \sqsubseteq \rho_f) - \Theta(\pi \sqsubseteq \rho_f \wedge \pi \neq \rho_f) \\
&= \Theta(\pi = \rho_f).
\end{aligned}$$

□

Lemma 3.3 *Let $|C| \geq m$. Any polynomial $\sum_{|\pi|\leq m} a_\pi\delta_\pi$ or $\sum_{|\pi|\leq m} b_\pi\varepsilon_\pi$ that vanishes under $[\![\ ]\!]_C$ is identically* 0.

Proof. Suppose $[\![\sum_{|\pi|\leq m} b_\pi\varepsilon_\pi]\!]_C = 0$. We show by induction on $\sqsubseteq$ that all coefficients b_π are 0. Let $|\sigma| \leq m$ and assume $b_\pi = 0$ for $\pi \sqsubseteq \sigma$, $\pi \neq \sigma$. We show that $b_\sigma = 0$ as well. Let $f : C^V \to R$ such that $\rho_f = \sigma$. Such an f exists by the assumption $|C| \geq m$.

$$\begin{aligned}
0 &= [\![\sum_{|\pi|\leq m} b_\pi\varepsilon_\pi]\!]_C(f) \\
&= \sum_{|\pi|\leq m} b_\pi\Theta(\pi = \sigma) \\
&= \sum_{\substack{|\pi|\leq m\\ \pi\sqsubseteq\sigma\\ \pi\neq\sigma}} b_\pi\Theta(\pi = \sigma) + \sum_{\substack{|\pi|\leq m\\ \pi=\sigma}} b_\pi\Theta(\pi = \sigma) + \sum_{\substack{|\pi|\leq m\\ \pi\not\sqsubseteq\sigma}} b_\pi\Theta(\pi = \sigma) \\
&= 0 + b_\sigma + 0 \\
&= b_\sigma.
\end{aligned}$$

The result for $\sum_{|\pi|\leq m} a_\pi\delta_\pi$ follows from this via (3.12) and (3.13). □

Theorem 3.4 (Completeness I) *For all C such that $|C| \geq n$,* ker $[\![\ ]\!]_C \subseteq I$.

Proof. Let $p \in R[\delta_A \mid A \subseteq V]$ such that $[\![p]\!]_C = 0$. We wish to show that $p \in I$. By Lemma 2.3, p is equivalent modulo I to a polynomial $\sum_\pi a_\pi\delta_\pi$, $a_\pi \in R$. By Theorem 3.1(i), $[\![\sum_\pi a_\pi\delta_\pi]\!]_C = 0$. By Lemma 3.3 with $m = n$, $\sum_\pi a_\pi\delta_\pi = 0$, therefore $p \in I$. □

3.5 Completeness for Deficient C

To prove completeness for deficient color classes, we need one more lemma.

Lemma 3.5 *If $|\pi| > m$ then $\varepsilon_\pi \in I_m$.*

Proof. Modulo I_m, ε_π is equivalent to

$$\varepsilon_\pi(1 - \prod_{|\rho|=m}(1-\delta_\rho)), \tag{3.14}$$

so it suffices to show that (3.14) is in I_m. In fact, it is in I. Let $|C| \geq n$. For all $f : V \to C$, by (3.11),

$$[\![1 - \prod_{|\rho|=m}(1-\delta_\rho)]\!]_C(f) \quad = \quad \Theta(|\rho_f| \leq m),$$

and since $|\pi| > m$,

$$\begin{aligned} [\![\varepsilon_\pi(1 - \textstyle\prod_{|\rho|=m}(1-\delta_\rho))]\!]_C(f) &= [\![\varepsilon_\pi]\!]_C(f) \cdot [\![1 - \textstyle\prod_{|\rho|=m}(1-\delta_\rho)]\!]_C(f) \\ &= \Theta(\pi = \rho_f) \cdot \Theta(|\rho_f| \leq m) \\ &= 0. \end{aligned}$$

As f was arbitrary, (3.14) is in ker $[\![\ ]\!]_C$. By Theorem 3.4, it is in I. □

Theorem 3.6 (Completeness II) *For all C such that $|C| \geq m$,* ker $[\![\ ]\!]_C \subseteq I_m$*.*

Proof. The proof is similar to Theorem 3.4. Let $p \in R[\delta_A \mid A \subseteq V]$ such that $[\![p]\!]_C = 0$. We wish to show that $p \in I_m$. Combining Lemma 2.3 with Lemma 3.5, p is equivalent modulo I_m to a polynomial $\sum_{|\pi|\leq m} b_\pi\varepsilon_\pi$, $b_\pi \in R$. By Theorem 3.1(ii), $[\![\sum_{|\pi|\leq m} b_\pi\varepsilon_\pi]\!]_C = 0$. By Lemma 3.3, $\sum_{|\pi|\leq m} b_\pi\varepsilon_\pi = 0$, therefore $p \in I_m$. □

The following are some consequences of the completeness theorem.

Corollary 3.7 *Let $m \leq n$ and let $|\pi| \geq m$. The following three expressions are all equivalent to δ_π modulo I_m:*

$$1 - \prod_{\substack{\pi\sqsubseteq\rho \\ |\pi|=m}}(1-\delta_\rho) \qquad 1 - \prod_{\substack{\pi\sqsubseteq\rho \\ |\pi|\leq m}}(1-\delta_\rho) \qquad 1 - \prod_{\substack{\pi\sqsubseteq\rho \\ |\pi|\leq m}}(1-\varepsilon_\rho). \tag{3.15}$$

Proof. Let $|C| = m$ and $f : V \to C$. Then

$$[\![\delta_\pi]\!]_C(f) \quad = \quad \Theta(\pi \sqsubseteq \rho_f). \tag{3.16}$$

Reasoning semantically, the three expressions reduce to

$$\begin{aligned}
[\![1-\prod_{\substack{\pi\sqsubseteq\rho\\|\rho|=m}}(1-\delta_\rho)]\!]_C(f) &= \Theta(\exists\rho\ \pi\sqsubseteq\rho\wedge|\rho|=m\wedge\rho\sqsubseteq\rho_f),\\
[\![1-\prod_{\substack{\pi\sqsubseteq\rho\\|\rho|\leq m}}(1-\delta_\rho)]\!]_C(f) &= \Theta(\exists\rho\ \pi\sqsubseteq\rho\wedge|\rho|\leq m\wedge\rho\sqsubseteq\rho_f),\\
[\![1-\prod_{\substack{\pi\sqsubseteq\rho\\|\rho|\leq m}}(1-\varepsilon_\rho)]\!]_C(f) &= \Theta(\exists\rho\ \pi\sqsubseteq\rho\wedge|\rho|\leq m\wedge\rho=\rho_f)\\
&= \Theta(\pi\sqsubseteq\rho_f\wedge|\rho_f|\leq m).
\end{aligned}$$

These are all equal to (3.16), since $|\rho_f|\leq m$. As f was arbitrary, the expressions (3.15) are equivalent to δ_π under the map $[\![\ ]\!]_C$. By completeness, they are equivalent modulo I_m. □

3.6 Dimensionality

We have actually shown

Theorem 3.8

(i) *The quotient rings* $R[\delta_A \mid A\subseteq V]/I$ *and* $R[\delta_A \mid A\subseteq V]/\ker\ [\![\ ]\!]_C$ *for* $|C|\geq n$ *are isomorphic* R*-modules of dimension* B_n*, the* n*th Bell number (number of partitions of a set of size* n*). The sets* $\{\varepsilon_\pi \mid \pi\in\Pi(V)\}$ *and* $\{\delta_\pi \mid \pi\in\Pi(V)\}$ *each form a basis.*

(ii) *The quotient rings* $R[\delta_A \mid A\subseteq V]/I_m$ *and* $R[\delta_A \mid A\subseteq V]/\ker\ [\![\ ]\!]_C$ *for* $|C|=m\leq n$ *are isomorphic* R*-modules of dimension* B_n^m*, the number of partitions of a set of size* n *with at most* m *partition elements*[1]*. The sets* $\{\varepsilon_\pi \mid \pi\in\Pi(V),\ |\pi|\leq m\}$ *and* $\{\delta_\pi \mid \pi\in\Pi(V),\ |\pi|\leq m\}$ *each form a basis.*

Proof. The statement (i) is a special case of (ii) with $m=n$.

For (ii), it follows from (3.12) and (3.13) that one of $\{\delta_\pi \mid \pi\in\Pi(V),\ |\pi|\leq m\}$ and $\{\varepsilon_\pi \mid \pi\in\Pi(V),\ |\pi|\leq m\}$ is a basis if and only if the other is, so we need only show the latter.

As argued in Theorem 3.6, every polynomial is equivalent modulo I_m to one of the form $\sum_{|\pi|\leq m} b_\pi\varepsilon_\pi$, thus $R[\delta_A \mid A\subseteq V]/I_m$ is an R-module of dimension at most B_n^m spanned by the ε_π, $|\pi|\leq m$. By Theorem 3.1, there is a homomorphism of R-modules

$$h: R[\delta_A \mid A\subseteq V]/I_m \quad\to\quad R[\delta_A \mid A\subseteq V]/\ker\ [\![\ ]\!]_C,$$

thus the dimension of $R[\delta_A \mid A\subseteq V]/\ker\ [\![\ ]\!]_C$ is bounded by the dimension of $R[\delta_A \mid A\subseteq V]/I_m$. By Lemma 3.3, the ε_π for $|\pi|\leq m$ are linearly independent

[1] This is $\sum_{m=0}^{n}\left\{ n \atop m \right\}$, where the $\left\{ n \atop m \right\}$ are Stirling numbers of the second kind [5].

modulo ker $[\![\ ]\!]_C$, thus the dimension of $R[\delta_A \mid A \subseteq V]/\ker [\![\ ]\!]_C$ as an R-module is at least B_n^m. Putting these observations together, we have

$$B_n^m \le \dim R[\delta_A \mid A \subseteq V]/\ker [\![\ ]\!]_C \le \dim R[\delta_A \mid A \subseteq V]/I_m \le B_n^m,$$

therefore all these inequalities are equalities, and the homomorphism h is an isomorphism. □

3.7 The Möbius Algebra

As mentioned in Section 2.3, the quotient $R[\delta_A \mid A \subseteq V]/I$ is isomorphic to the semigroup algebra of $\Pi(V)$, the upper semilattice of partitions of V, over R. This is known as the *Möbius algebra* of the semilattice $\Pi(V)$ [6]. Informally, it consists of linear combinations of elements of $\Pi(V)$ with coefficients in R. Formally, it can be characterized in two ways: (i) as a quotient ring $R[\delta_\pi \mid \pi \in \Pi(V)]/J$, where δ_π for $\pi \in \Pi(V)$ is a set of indeterminates and J is the ideal generated by

$$\delta_\pi\delta_\rho - \delta_{\pi\sqcup\rho} \qquad \delta_\iota - 1, \tag{3.17}$$

where ι is the identity partition $\{\{u\} \mid u \in V\}$; or (ii) as the coproduct of R and $\mathbb{Z}_k[\Pi(V)]$ in the category of commutative rings of characteristic k, where k is the characteristic of R and $\mathbb{Z}_k[\Pi(V)]$ is the image of $\Pi(V)$ under the left adjoint of the forgetful functor that takes a ring to its multiplicative monoid.

Theorem 3.9 $R[\delta_A \mid A \subseteq V]/I \;\cong\; R[\delta_\pi \mid \pi \in \Pi(V)]/J.$

Proof. Let $h(\delta_A) \stackrel{\text{def}}{=} \delta_{\rho_A}$, where ρ_A is the partition generated by A. Let $g(\delta_\pi) = \prod_{A\in\pi}\delta_A$. The maps h and g extend uniquely to homomorphisms

$$\begin{aligned} h &: R[\delta_A \mid A \subseteq V] \;\to\; R[\delta_\pi \mid \pi \in \Pi(V)]/J \\ g &: R[\delta_\pi \mid \pi \in \Pi(V)] \;\to\; R[\delta_A \mid A \subseteq V]/I, \end{aligned}$$

and one can show without difficulty that $I \subseteq \ker h$ and $J \subseteq \ker g$, therefore h and g induce homomorphisms between the two quotient rings in the statement of the lemma, and that the induced homomorphisms are inverses. □

This result says that we can take (3.17) as an alternative axiomatization.

4 Indefinite Summation

In this section we introduce the indefinite summation operator and its axiomatization.

For $\pi \in \Pi(V)$ and $x \in V$, we say x is *isolated* in π if $\{x\} \in \pi$. For any π, let $\pi|x$ be the partition obtained from π by replacing the unique set $A \in \pi$ containing x with the sets $A - \{x\}$ (if it is nonempty) and $\{x\}$. The partition

$\pi|x$ is the coarsest refinement of π in which x is isolated. If x is isolated in π, then $\pi|x = \pi$.

Let $\mathbb{Q}$ be the field of rational numbers. Let X be a new indeterminate. For $x \in V$, define the linear operator

$$\sum_x : \mathbb{Q}[X, \delta_A \mid A \subseteq V]/I \quad \to \quad \mathbb{Q}[X, \delta_A \mid A \subseteq V]/I$$

to be the unique $\mathbb{Q}[X]$-module homomorphism such that

$$\sum_x \delta_\pi \quad \stackrel{\text{def}}{=} \quad \begin{cases} \delta_{\pi|x}, & \text{if } x \text{ is not isolated in } \pi \\ \delta_\pi X, & \text{otherwise.} \end{cases} \tag{4.1}$$

By Theorem 3.8(i), this uniquely determines the map $\sum_x$ on all elements of $\mathbb{Q}[X, \delta_A \mid A \subseteq V]/I$ (the R in Theorem 3.8(i) is $\mathbb{Q}[X]$). The linear operator $\sum_x$ is called an *indefinite summation operator*. Intuitively, $\sum_x$ behaves like a summation operator with indefinite bound X, which will be interpreted as the number of colors.

To formulate soundness and completeness in the presence of the indefinite summation operators, we must augment the language of polynomial expressions with new unary operator symbols $\sum_x$, one for each $x \in V$. Expressions in this new language are called *extended polynomial expressions*. Modulo the ring axioms and $\mathbb{Q}[X]$-linearity for $\sum_x$ (that is, $\sum_x(ap + bq) = a\sum_x p + b\sum_x q$ for $a, b \in \mathbb{Q}[X]$), the set of extended polynomial expressions forms a $\mathbb{Q}[X]$-algebra with operators $\sum_x$, which we denote by $\mathbb{Q}[\Sigma, X, \delta_A \mid A \subseteq V]$.

An *ideal* in this algebra is a ring ideal J such that if $p \in J$, then $\sum_x p \in J$. Ideals are the kernels of $\mathbb{Q}[X]$-algebra homomorphisms that also preserve $\sum_x$. Let $\langle A \rangle$ denote the ideal generated by the set A.

Abbreviate $A - \{x\}$ by $A - x$. Let $\widehat{I}$ be the ideal in $\mathbb{Q}[\Sigma, X, \delta_A \mid A \subseteq V]$ generated by I and the following extended expressions.

(4.2) $(\sum_x pq) \; - \; p\sum_x q$, if p does not involve x, that is, if the expression p does not contain any δ_A with $x \in A$;

(4.3) $(\sum_x \delta_A) \; - \; \delta_{A-x}, \quad x \in A, \; |A| \geq 2$;

(4.4) $(\sum_x 1) \; - \; X$.

These expressions correspond to equational axioms

- $\sum_x pq = p\sum_x q$ for p not involving x,
- $\sum_x \delta_A = \delta_{A-x}$ for $x \in A$ and $|A| \geq 2$, and
- $\sum_x 1 = X$,

respectively.

Theorem 4.1 *The operators $\sum_x$ on $\mathbb{Q}[X, \delta_A \mid A \subseteq V]/I$ defined by* (4.1) *satisfy the equations corresponding to* (4.2)–(4.4)*. Conversely,* (4.2)–(4.4) *and linearity uniquely determine $\sum_x$ on $\mathbb{Q}[X, \delta_A \mid A \subseteq V]/I$.*

Proof. To show (4.2), suppose p does not involve x. By Lemma 2.3, write $p = \sum_\pi a_\pi \delta_\pi$ and $q = \sum_\rho b_\rho \delta_\rho$. Reasoning modulo I, we get

$$\sum_x pq \quad = \quad \sum_x (\sum_\pi a_\pi \delta_\pi)(\sum_\rho b_\rho \delta_\rho) \quad = \quad \sum_{\pi,\rho} a_\pi b_\rho \sum_x \delta_{\pi \sqcup \rho}. \tag{4.5}$$

Now consider any $\delta_{\pi\sqcup\rho}$ in the sum. Since p did not involve x, x is isolated in π, therefore x is isolated in $\pi \sqcup \rho$ iff it is isolated in ρ. It follows that $(\pi \sqcup \rho)|x = \pi \sqcup (\rho|x)$, so

$$\begin{aligned}
\sum_x \delta_{\pi\sqcup\rho} \quad &= \quad \begin{cases} \delta_{(\pi\sqcup\rho)|x}, & \text{if } \pi \sqcup \rho \neq (\pi \sqcup \rho)|x \\ \delta_{\pi\sqcup\rho} X, & \text{otherwise} \end{cases} \\
&= \quad \begin{cases} \delta_\pi \delta_{\rho|x}, & \text{if } \rho \neq \rho|x \\ \delta_\pi \delta_\rho X, & \text{otherwise} \end{cases} \\
&= \quad \delta_\pi \sum_x \delta_\rho.
\end{aligned}$$

Substituting this in (4.5), we obtain

$$\sum_x pq \quad = \quad \sum_{\pi,\rho} a_\pi b_\rho \delta_\pi \sum_x \delta_\rho \quad = \quad p \sum_x q.$$

For (4.3), let $x \in A$, $|A| \geq 2$, and let π be the partition generated by A. Then $\pi|x$ is the partition generated by $A - \{x\}$. Thus

$$\sum_x \delta_A \quad = \quad \sum_x \delta_\pi \quad = \quad \delta_{\pi|x} \quad = \quad \delta_{A-\{x\}}.$$

Finally, for (4.4),

$$\sum_x 1 \quad = \quad \sum_x \delta_\iota \quad = \quad \delta_\iota X \quad = \quad X.$$

Conversely, (4.1) follows from (4.2)–(4.4). If $\pi = \pi|x$, then δ_π contains the factor δ_x, which can be removed since $1 - \delta_x \in I$; then (4.2) can be applied, followed by (4.4). If $\pi \neq \pi|x$, so that the A in π containing x is of size at least 2, then (4.2) can be applied, followed by (4.3). $\square$

We have essentially shown

Corollary 4.2 *The structures*

$$\mathbb{Q}[X, \delta_A \mid A \subseteq V]/I \quad \textit{and} \quad \mathbb{Q}[\Sigma, X, \delta_A \mid A \subseteq V]/\widehat{I}$$

are isomorphic as $\mathbb{Q}[X]$-algebras with operators $\sum_x$.

4.1 Soundness and Completeness

We now extend the semantic interpretation $[\![\ ]\!]_C$ to extended expressions and prove soundness and completeness over these interpretations. For $x \in V$ and $c \in C$, let

$$f\,[x/c]\,(y) \quad \stackrel{\text{def}}{=} \quad \begin{cases} f(y), & \text{if } y \neq x, \\ c, & \text{if } y = x. \end{cases}$$

Thus $f\,[x/c]$ takes the same value as f on all inputs except x, on which it takes the value c. The operator $[x/c]$ that takes f to $f\,[x/c]$ is called a *rebinding operator* because it rebinds x to the value c.

In addition to (3.1) for $[\![\delta_A]\!]_C(f)$, define

$$[\![X]\!]_C(f) \quad \stackrel{\text{def}}{=} \quad |C| \tag{4.6}$$

$$[\![\textstyle\sum_x p]\!]_C(f) \quad \stackrel{\text{def}}{=} \quad \displaystyle\sum_{c \in C} [\![p]\!]_C(f\,[x/c]\,). \tag{4.7}$$

Thus $[\![\sum_x p]\!]_C$, given a coloring f, sums the value of the expression $[\![p]\!]_C$ over all colorings obtained by recoloring x and leaving the colors of the other elements the same.

Lemma 4.3 *For a polynomial* p, *if* p *does not involve* x, *then* $[\![p]\!]_C(f\,[x/c]\,) = [\![p]\!]_C(f)$.

Proof. It suffices to show the result for $p = \delta_A$, where $x \notin A$. In this case the result reduces to the observation that $\rho_A \sqsubseteq \rho_{f\,[x/c]}$ iff $\rho_A \sqsubseteq \rho_f$. This is true because x is isolated in ρ_A, so its color does not matter in determining whether $\rho_A \sqsubseteq \rho_f$. $\square$

Theorem 4.4 (Soundness II) *Linearity and the axioms* (4.2)–(4.4) *are sound with respect to all interpretations* $[\![\ ]\!]_C$. *The axiom* $X = m$ *is sound if* $|C| = m$. *In other words, for* $|C| = m$, $\langle \widehat{I},\ X - m\rangle \subseteq \ker\ [\![\ ]\!]_C$.

Proof. The soundness of linearity is a straightforward consequence of the linearity of $[\![\ ]\!]_C$. For the others, let $f : C^V \to \mathbb{Q}$ be arbitrary. For (4.2) we use Lemma 4.3. Assume that p does not involve x.

$$\begin{aligned}
[\![\textstyle\sum_x pq]\!]_C(f) \quad &= \quad \displaystyle\sum_{c \in C} [\![pq]\!]_C(f\,[x/c]\,) \quad = \quad \sum_{c \in C} [\![p]\!]_C(f\,[x/c]\,)\,[\![q]\!]_C(f\,[x/c]\,) \\
&= \quad \sum_{c \in C} [\![p]\!]_C(f)\,[\![q]\!]_C(f\,[x/c]\,) \quad = \quad [\![p]\!]_C(f) \sum_{c \in C} [\![q]\!]_C(f\,[x/c]\,) \\
&= \quad [\![p]\!]_C(f)\,[\![\textstyle\sum_x q]\!]_C(f) \quad = \quad [\![p \textstyle\sum_x q]\!]_C(f).
\end{aligned}$$

For (4.3), let $x \in A$, $|A| \geq 2$. We abbreviate $A - \{x\}$ by $A - x$.

$$\begin{aligned}
[\![\textstyle\sum_x \delta_A]\!]_C(f) &= \sum_{c \in C} [\![\delta_A]\!]_C(f[x/c]) \;=\; \sum_{c \in C} \Theta(\rho_A \sqsubseteq \rho_{f[x/c]}) \\
&= \sum_{c \in C} \Theta(\forall y, z \in A - x \ f(y) = f(z) = c) \\
&= \Theta(\forall y, z \in A - x \ f(y) = f(z)) \;=\; [\![\delta_{A-x}]\!]_C(f).
\end{aligned}$$

For (4.4),

$$[\![\textstyle\sum_x 1]\!]_C(f) \;=\; \sum_{c \in C} [\![1]\!]_C(f[x/c]) \;=\; \sum_{c \in C} 1 \;=\; |C| \;=\; [\![X]\!]_C(f).$$

If $|C| = m$, then the soundness of $X = m$ is immediate from (4.6). $\square$

The following theorem asserts that the axioms (4.2)–(4.4) (equivalently, (4.1)) and linearity, in conjunction with the complete axiomatization for ordinary (nonextended) polynomial expressions established in Section 3, are complete for the equational theory of extended expressions with respect to all interpretations $[\![\ ]\!]_C$. That is, any pair of extended expressions that are equal under all interpretations $[\![\ ]\!]_C$ are provably equal in equational logic from the axioms. Equivalently, any extended expression that vanishes under all interpretations $[\![\ ]\!]_C$ is provably equal to 0.

Theorem 4.5 (Completeness III) $\bigcap_C \ker [\![\ ]\!]_C \subseteq \widehat{I}$.

Proof. Modulo $\widehat{I}$, every extended expression p can be reduced to a $\sum_x$-free expression p' by starting from the innermost occurrences of $\sum_x$ and working outward, eliminate each occurrence by an application of linearity and (4.2)–(4.4). In turn, by Lemma 2.3, p' is equivalent modulo I to an expression $\sum_\pi a_\pi \delta_\pi$, where $a_\pi = a_\pi(X) \in \mathbb{Q}[X]$.

If $[\![p]\!]_C = 0$ for all C, then by soundness (Theorem 4.4), $[\![\sum_\pi a_\pi \delta_\pi]\!]_C = 0$ for all C. Interpreting under $[\![\ ]\!]_C$ with $|C| = k \geq n$, for any $f : V \to C$,

$$0 \;=\; [\![\sum_\pi a_\pi \delta_\pi]\!]_C(f) \;=\; \sum_\pi a_\pi(k) [\![\delta_\pi]\!]_C(f) \;=\; [\![\sum_\pi a_\pi(k) \delta_\pi]\!]_C(f).$$

By Lemma 3.3, all $a_\pi(k) = 0$. As this is true for arbitrarily large values of k, the polynomials $a_\pi(X)$ are all identically 0. $\square$

The following theorem asserts that the axioms mentioned in Theorem 4.5 along with the polynomial $X - m$ and, if $m \leq n$, the polynomial (3.9) are complete for the equational theory of extended expressions with respect to the interpretation $[\![\ ]\!]_C$ for $|C| = m$.

Let $\widehat{I}_m$ be the ideal in $\mathbb{Q}[\Sigma, X, \delta_A \mid A \subseteq V]$ generated by I_m and (4.2)–(4.4).

Theorem 4.6 (Completeness IV)

(i) $\ker [\![\ ]\!]_C \subseteq \langle \widehat{I}_m,\ X - m\rangle$ *for* $m \le n$*;*

(ii) $\ker [\![\ ]\!]_C \subseteq \langle \widehat{I},\ X - m\rangle$ *for* $m \ge n$*.*

Proof. For (i), as in the proof of Theorem 4.5, any extended expression p is equivalent modulo $\widehat{I}_m$ to an expression $\sum_{\pi \le m} a_\pi \delta_\pi$, where $a_\pi = a_\pi(X) \in \mathbb{Q}[X]$. Again by Lemma 3.3, if $[\![p]\!]_C = 0$, then all $a_\pi(m) = 0$, thus all $a_\pi(X)$ vanish modulo $X - m$, therefore so does $\sum_{\pi \le m} a_\pi \delta_\pi$.

The proof of (ii) is similar. □

We have actually shown that the operator $\sum_x$ on $\mathbb{Q}[X, \delta_A \mid A \subseteq V]/I$ commutes with the linear operator

$$S_x^C : (C^V \to \mathbb{Q}) \quad \to \quad (C^V \to \mathbb{Q}),$$

defined for $F : C^V \to \mathbb{Q}$ by

$$S_x^C(F)(f) \quad \stackrel{\text{def}}{=} \quad \sum_{c \in C} F(f[x/c])$$

under the interpretation $[\![\ ]\!]_C$.

4.2 Deficient Color Classes

Curiously, $\sum_x$ cannot be defined on $\mathbb{Q}[X, \delta_A \mid A \subseteq V]/I_m$ to satisfy linearity and (4.2)–(4.4), since these axioms are strictly stronger than I_m. However, it can be defined trivially on $\mathbb{Q}[\Sigma, X, \delta_A \mid A \subseteq V]/\widehat{I}_m$. The definition is the obvious one: the operator $\sum_x$ maps the expression p to the expression $\sum_x p$.

For $m < n$, the ideal $\widehat{I}_m$ in $\mathbb{Q}[\Sigma, X, \delta_A \mid A \subseteq V]$ is strictly larger than $\langle I_m \rangle$, the ideal generated by I_m. Specifically, let $(X)_m$ denote the falling factorial power $X(X-1)\cdots(X-m+1)$. The ideal $\widehat{I}_m$ contains $(X-1)_m$ (Lemma 4.7), but $\langle I_m \rangle$ does not. This makes intuitive sense, since it is consistent with the restriction $|C| \le m$. Note also that the polynomial $(X)_m$ is the chromatic polynomial of the complete graph with m vertices (see Section 5).

The main result of this section (Theorem 4.12) is that $\widehat{I}_m$ is exactly the intersection of the kernels of $[\![\ ]\!]_C$ for $|C| \le m$.

Lemma 4.7 *For* $m < n$*,* $(X-1)_m \in \widehat{I}_m$*.*

Proof. Suppose $m < n$. Let $U \subseteq V$ such that $|U| = m + 1$. Let

$$\begin{aligned} D &\stackrel{\text{def}}{=} \{\delta_\pi \mid |\pi| = n-1,\ \text{all } x \notin U \text{ are isolated in } \pi\} \\ &= \{\delta_{xy} \mid x, y \in U\} \pmod{I}. \end{aligned}$$

If $|\rho| = m$, let

$$\sigma \quad = \quad \{A \cap U \mid A \in \rho,\ A \cap U \neq \varnothing\} \cup \{\{x\} \mid x \notin U\}.$$

Then $\sigma \sqsubseteq \rho$ and σ has at least one nonsingleton subset of U, since $|U| > m$. By further refining σ if necessary, we can obtain a $\pi \in D$ such that $\pi \sqsubseteq \rho$. Thus every ρ with $|\rho| = m$ has a refinement in D.

Moreover, if $\pi \sqsubseteq \rho$, then $(1-\delta_\pi)(1-\delta_\rho) = 1-\delta_\pi$, since

$$(1-\delta_\pi)(1-\delta_\rho) \quad = \quad 1-\delta_\pi-\delta_\rho+\delta_{\pi\sqcup\rho} \quad = \quad 1-\delta_\pi.$$

It follows from these two facts that

$$\prod_{\pi\in D}(1-\delta_\pi) \quad = \quad \prod_{\pi\in D}(1-\delta_\pi)\prod_{|\rho|=m}(1-\delta_\rho) \quad \in \quad I_m. \tag{4.8}$$

Now let $z \in U$ be arbitrary. Applying all the indefinite summation operators $\sum_x$ for $x \in U$ except z (we abbreviate this by $\sum_{U-z}$),

$$\sum_{U-z}\prod_{\pi\in D}(1-\delta_\pi) \quad = \quad \sum_{U-z}\prod_{x,y\in U}(1-\delta_{xy}). \tag{4.9}$$

The expression (4.9) is equivalent modulo I to a polynomial in $\mathbb{Q}[X]$, since it contains only singletons δ_z, which modulo I are 1.

We claim that this polynomial is exactly $(X-1)_m$. The chromatic polynomial on an $(m+1)$-element complete graph is

$$(X)_{m+1} \quad = \quad \sum_U\prod_{x,y\in U}(1-\delta_{xy}),$$

and

$$\sum_U\prod_{x,y\in U}(1-\delta_{xy}) \quad = \quad \sum_z\sum_{U-z}\prod_{x,y\in U}(1-\delta_{xy}) \quad = \quad X(\sum_{U-z}\prod_{x,y\in U}(1-\delta_{xy}))$$

since (4.9) is a polynomial in $\mathbb{Q}[X]$, so

$$(X-1)_m \quad = \quad \frac{(X)_{m+1}}{X} \quad = \quad \sum_{U-z}\prod_{\pi\in D}(1-\delta_\pi).$$

This polynomial is in $\widehat{I}_m$ by (4.8), since $I_m \subseteq \widehat{I}_m$ and $\widehat{I}_m$ is closed under $\sum_x$. $\square$

For $m \leq n$, let

$$p_m \quad \stackrel{\text{def}}{=} \quad \prod_{|\rho|=m}(1-\delta_\rho),$$

the polynomial (3.9) such that $I_m = \langle I,\ p_m\rangle$. For $U \subseteq V$, let

$$q_U \stackrel{\text{def}}{=} \prod_{y,z\in U}(1-\delta_{yz}) \qquad\qquad Q_m \stackrel{\text{def}}{=} 1-\prod_{|U|=m+1}(1-q_U).$$

Note that $[\![q_U]\!]_C(f) = \Theta(\forall y,z \in U\ f(y) \neq f(z))$.

Lemma 4.8 *The polynomials p_m and Q_m are equivalent modulo I.*

Proof. By Theorem 3.4, it suffices to show that they are equivalent under any interpretation $[\![\ ]\!]_C$. By (3.11), $[\![p_m]\!]_C(f) = \Theta(|\rho_f| > m)$. But Q_m has the same interpretation: for any $f : V \to C$,

$$\begin{aligned}
[\![Q_m]\!]_C(f) &= 1 - \prod_{|U|=m+1} (1 - \Theta(\forall y, z \in U\ f(y) \neq f(z))) \\
&= 1 - \Theta(\forall U\ |U| = m+1 \Rightarrow \exists y, z \in U\ f(y) = f(z)) \\
&= \Theta(\exists U\ |U| = m+1 \wedge \forall y, z \in U\ f(y) \neq f(z)) \\
&= \Theta(|\rho_f| > m).
\end{aligned}$$

□

Lemma 4.9 $I_m = \langle I,\ q_U \mid |U| = m+1 \rangle$.

Proof. The forward inclusion follows from Lemma 4.8 and the observation that $Q_m \in \langle q_U \mid |U| = m+1 \rangle$.

For the reverse inclusion, we show that $q_U \in I_m$ whenever $|U| = m+1$. If $|C| = m$, then

$$[\![q_U]\!]_C(f) \quad = \quad \Theta(\forall y, z \in U\ f(y) \neq f(z)) \quad = \quad 0,$$

thus $[\![q_U]\!]_C = 0$, and $q_U \in \ker [\![\ ]\!]_C \subseteq I_m$ by Theorem 3.6. □

Lemma 4.10 *Let $|U| = m$ and $x \notin U$. Write q_{U+x} for $q_{U \cup \{x\}}$. Modulo $\widehat{I}$,*

$$\sum_x q_{U+x} \quad = \quad (X - m) q_U.$$

Proof. Applying (4.2)–(4.4) to eliminate the summation from the expression $\sum_x q_{U+x}$, we obtain a polynomial $r(X)$ with coefficients in $\mathbb{Q}[X]$ that is equivalent modulo $\widehat{I}$ to $\sum_x q_{U+x}$. For any C and $f : V \to C$,

$$\begin{aligned}
[\![r(X)]\!]_C(f) &= [\![\textstyle\sum_x q_{U+x}]\!]_C(f) \\
&= \sum_{c \in C} [\![q_{U+x}]\!]_C(f[x/c]) \\
&= \sum_{c \in C} \Theta(\forall y, z \in U \cup \{x\}\ f[x/c](y) \neq f[x/c](z)) \\
&= \sum_{c \in C} \Theta(\forall y, z \in U\ f(y) \neq f(z) \wedge f(y) \neq c) \\
&= \left(\sum_{c \in C} \Theta(\forall y \in U\ f(y) \neq c) \right) \Theta(\forall y, z \in U\ f(y) \neq f(z)) \\
&= (|C| - m) \cdot \Theta(\forall y, z \in U\ f(y) \neq f(z)) \\
&= [\![X - m]\!]_C(f)\, [\![q_U]\!]_C(f) \\
&= [\![(X - m) q_U]\!]_C(f).
\end{aligned}$$

As f was arbitrary,

$$[\![r(|C|)]\!]_C \quad = \quad [\![r(X)]\!]_C \quad = \quad [\![(X-m)q_U]\!]_C \quad = \quad [\![(|C|-m)q_U]\!]_C.$$

This is true for arbitrarily large C, thus by Theorem 3.4, $r(n)$ and $(n-m)q_U$ are equivalent modulo I for arbitrarily large n. The polynomial coefficients must be equal, therefore $r(X)$ and $(X-m)q_U$ are equivalent modulo $\widehat{I}$. $\square$

Lemma 4.11 *If $k \leq m-1$, then $\widehat{I}_{m-1} \subseteq \langle \widehat{I}_m,\ X-k\rangle$.*

Proof. Let $|U| = m$ and $x \notin U$. From Lemma 4.9, we have $q_{U+x} \in I_m$. By Lemma 4.10, $(X-m)q_U = \sum_x q_{U+x} \in \widehat{I}_m$. Since $k \neq m$, $q_U \in \langle \widehat{I}_m,\ X-k\rangle$. As U was arbitrary, by Lemma 4.9, $p_{m-1} \in \langle \widehat{I}_m,\ X-k\rangle$. Since $\widehat{I}_{m-1} = \langle \widehat{I}, p_{m-1}\rangle$ and $\widehat{I} \subseteq \widehat{I}_m$, the result follows. $\square$

Theorem 4.12 *For $m < n$, $\widehat{I}_m = \bigcap_{|C|\leq m} \ker [\![\ ]\!]_C$.*

Proof. By Theorem 4.6(i), for $|C| = k \leq m$, $\ker [\![\ ]\!]_C = \langle \widehat{I}_k,\ X-k\rangle$. By $m-k$ applications of Lemma 4.11, we have $\langle \widehat{I}_k,\ X-k\rangle \subseteq \langle \widehat{I}_m,\ X-k\rangle$; but since $\widehat{I}_m \subseteq \widehat{I}_k$, they are equal. Thus

$$\bigcap_{|C|\leq m} \ker [\![\ ]\!]_C \quad = \quad \bigcap_{k\leq m} \langle \widehat{I}_k,\ X-k\rangle \quad \subseteq \quad \bigcap_{k\leq m} \langle \widehat{I}_m,\ X-k\rangle.$$

Now using an argument from [16, p. 138], since the $X-k$ for $k \leq m$ are relatively prime, we have

$$\bigcap_{k\leq m} \langle \widehat{I}_m,\ X-k\rangle \quad = \quad \langle \widehat{I}_m,\ \prod_{k\leq m}(X-k)\rangle,$$

which by Lemma 4.7 is just $\widehat{I}_m$. $\square$

5 An Application

In this section we illustrate the use of the calculus to derive the chromatic polynomial of an undirected graph $G = (V, E)$. This is the polynomial function $\chi_G(X)$ whose value $\chi_G(k)$ on an integer k is the number of vertex colorings of G with k colors such that no pair of adjacent vertices receive the same color.

The chromatic polynomial is normally defined inductively in terms of edge contractions and deletions:

$$\chi_G(X) \quad = \quad \chi_{G-e}(X) - \chi_{G/e}(X), \tag{5.1}$$

where $G-e$ denotes G with the edge e deleted and G/e denotes G with the edge e deleted and its endpoints identified. The basis is $\chi_G(X) = X^k$ for a set of k vertices with no edges. Intuitively, (5.1) captures the idea that the number of proper colorings of G is the number of proper colorings of G without the constraint e less the number of colorings that violate only the constraint e. This equation provides a recursive method, albeit very inefficient, for computing the chromatic polynomial of a graph.

There is however a closed form of the chromatic polynomial, namely

$$\chi_G(X) \quad = \quad \sum_{F \subseteq E} (-1)^{|F|} X^{|c(F)|},$$

where the summation is over all subsets F of edges and $c(F) \in \Pi(V)$ is the set of connected components of the subgraph (V, F), including isolated vertices. Note that $c(F)$ is just the partition of V generated by F in the sense of Section 2.2.

Here is how we would derive this symbolically in our calculus.

Lemma 5.1

(i) *For* $A \subseteq V$, $A \neq \varnothing$, $\sum_A \delta_A = X$*;*

(ii) *For* $\pi \in \Pi(V)$, $\sum_V \delta_\pi = X^{|\pi|}$.

Proof. (i) By induction. If $|A| \leq 1$, $\sum_A \delta_A = \sum_A 1 = X$ by (4.4). If $|A| \geq 2$, let $x \in A$. By (4.3), $\sum_A \delta_A = \sum_{A-x} \sum_x \delta_A = \sum_{A-x} \delta_{A-x} = X$.

(ii) By (4.3), (4.4), and (i),

$$\sum_V \delta_\pi \quad = \quad \sum_V \prod_{A \in \pi} \delta_A \quad = \quad \prod_{A \in \pi} \sum_A \delta_A \quad = \quad \prod_{A \in \pi} X \quad = \quad X^{|\pi|}.$$

□

For each edge $e \in V$, the Kronecker delta δ_e selects those colorings for which the endpoints of e receive the same color. Thus the expression

$$\prod_{e \in E} (1 - \delta_e),$$

interpreted as a map $C^V \to \{0, 1\}$, takes value 1 on $f : V \to C$ if f is a proper coloring of G, 0 otherwise. Summing over all possible colorings,

$$\chi_G \quad = \quad \sum_V \prod_{e \in E} (1 - \delta_e).$$

Now using the various axioms and Lemma 5.1,

$$\begin{aligned}
\chi_G \quad &= \quad \sum_V \prod_{e \in E} (1 - \delta_e) \quad = \quad \sum_V \sum_{F \subseteq E} (-1)^{|F|} \prod_{e \in F} \delta_e \\
&= \quad \sum_V \sum_{F \subseteq E} (-1)^{|F|} \delta_F \quad = \quad \sum_{F \subseteq E} (-1)^{|F|} \sum_V \delta_F \\
&= \quad \sum_{F \subseteq E} (-1)^{|F|} \sum_V \delta_{c(F)} \quad = \quad \sum_{F \subseteq E} (-1)^{|F|} X^{|c(F)|}.
\end{aligned}$$

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