ISOPERIMETRIC CONSTANTS AND SELF-AVOIDING WALKS AND POLYGONS ON HYPERBOLIC COXETER GROUPS

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ISOPERIMETRIC CONSTANTS AND SELF-AVOIDING WALKS AND POLYGONS ON HYPERBOLIC COXETER GROUPS

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We study isoperimetric constants of and self-avoiding walks (SAWs) and self-avoiding polygons (SAPs) on Cayley graphs of hyperbolic Coxeter groups. These graphs are tilings of the hyperbolic plane. We calculate the isoperimetric constants of most rank three hyperbolic Coxeter groups and give an estimate for the remainder. We prove that for all but a few classes of these groups there are exponentially fewer k-step SAPs than there are k-step SAWs. We prove that this is also true of any nontrivial free product. Additionally we show that there are exponentially more k-step SAWs on a free group F_n (resp. a "free" Coxeter group T_d) than on any strict subgroup of F_n (resp. T_d).

BIOGRAPHICAL SKETCH

Jason Bode grew up in Lynden, a small town in the northwest corner of Washington, barely 3 miles from Canada. Despite being a nerd, as his sisters will attest, he enjoyed soccer, regularly challenged his Grandpa at pool and played in softball leagues. However, he solidified his status as a nerd while in high school as his interests turned towards mathematics and computing.

Jason pursued those interests at Calvin College, receiving liberal arts degrees in both mathematics and computer science while learning to love teaching through tutoring. During his years at Calvin, he made numerous lifelong friends and spent many hours playing ultimate frisbee and taking his roommates' quotes out of context.

After waffling on whether to focus on computers or math, Jason chose to pursue mathematics in graduate school at Cornell University. His first few years were filled with courses, homework, studying and, in spare time (and some that was not), friends, playing cards, softball and geocaching. He worked under Kenneth Brown with the help of Laurent Saloff-Coste and David Henderson, who also assisted Jason in exploring the art of education.

Through a Bible study, he met his soon-to-be wife Sallie, a veterinarian and lovely South Carolina belle. Naturally her cat (now theirs) is named Rhett Butler, who alternates between attempting to help with Jason's research and trying to get Jason to feed him.

Much to Sallie's delight, Jason's first position is at Lander University in Greenwood, South Carolina.

Soli Deo gloria

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Chapter 1

Isoperimetric Constants

1.1 Introduction

Isoperimetric constants of graphs have long been studied, yet there are few explicitly calculated examples. In this chapter we derive isoperimetric constants for the Cayley graphs of many rank three hyperbolic Coxeter groups and provide a lower bound for the remainder.

A Coxeter group W is a group with a presentation

$$W = \langle a_1, a_2, \ldots, a_d \mid (g_i g_j)^{m_{ij}} \rangle$$

satisfying $m_{ii} = 1$, implying the generators are involutions, and $2 \leq m_{ij} = m_{ji} \leq \infty$ (where $m_{ij} = \infty$ indicates no relator). Common examples of Coxeter groups are Euclidean and hyperbolic reflection groups. For more information on Coxeter groups, see [2] or [5].

The Cayley graph of a group G depends on the generating set, so throughout we will be considering a fixed choice of generating set. To create the Cayley graph of the group G with presentation

$$G = \langle a_i \mid r_i \rangle,$$

begin with a set of vertices $V = G$ (as a set), or in other words a vertex of the graph for each element of the group. Place an edge from g_1 to g_2 for each generator a_i where $g_2 = g_1 a_i$. Note that if both a and a^{-1} are in the generating set, there will be (at least) two edges between the vertices g and ga. If a generator is of order 2, i.e. $a = a^{-1}$, we will by convention identify the two edges from g to ga

with labels a and a^{-1}. Note that if there are repeated generators then the Cayley graph may be a multigraph. Most of the Cayley graphs considered below satisfy the properties that each is regular and there is at most one edge between a pair of vertices. We also are able to accurately describe Cayley graphs of Coxeter groups below by utilizing Tits' solution to the word problem for Coxeter groups (see [2], [5] or [11]).

Another option for visualizing Cayley graphs of the Coxeter groups that we consider is the observation that these graphs are dual graphs of particular tessellations of the hyperbolic plane by triangles. We utilize this in Section 1.3 for the Cayley graphs of the hyperbolic groups $W_{m,n,p}$.

For the study of isoperimetric constants, we begin with any graph $G = (V, E)$. Let $[x, y]$ represent any (unoriented) edge from x to y. For any set $K \subseteq V$, define

$$\begin{aligned} E(K) &:= \{[x,y] \in E \mid x, y \in K\}, \\ E^*(K) &:= \{[x,y] \in E \mid x \in K \text{ or } y \in K\}, \\ \partial K &:= E^*(K) \setminus E(K) \\ G(K) &:= (K, E(K)). \end{aligned}$$

Using these sets we define the following isoperimetric constants:

$$I_G := \liminf_{|K|\to\infty} \left\{ \frac{|\partial(K)|}{|K|} \mid K \subset V \text{ finite}, G(K) \text{ connected} \right\} \tag{1.1}$$

$$\beta(G) := \liminf_{|K|\to\infty} \left\{ \frac{|K|}{|E(K)|} \mid K \subset V \text{ finite}, G(K) \text{ connected} \right\} \tag{1.2}$$

$$\delta(G) := \limsup_{|K|\to\infty} \left\{ \frac{|K|}{|E^*(K)|} \mid K \subset V \text{ finite}, G(K) \text{ connected} \right\} \tag{1.3}$$

When G is transitive, it is shown in [1] that

$$I_G := \inf_{K \text{ finite}} \left\{ \frac{|\partial(K)|}{|K|} \mid K \subset V,\ G(K) \text{ connected} \right\}$$

Note further that the assumption of connectivity is easily dropped because breaking up a disconnected set into connected components will yield one component K_* with

$$\frac{|\partial(K_*)|}{|K_*|} \leq \frac{|\partial(K)|}{|K|}$$

When G is regular, we write d for the degree of G and, as in [3], we have

$$\beta(G) = \frac{2}{d - I_G} \qquad \text{and} \qquad \delta(G) = \frac{2}{d + I_G} \tag{1.4}$$

1.2 Calculating Isoperimetric Constants

1.2.1 "Free" Coxeter Groups

First, we begin with all Coxeter groups whose Cayley graphs are trees. The Cayley graph of the group

$$T_d = \mathbb{Z}/2\mathbb{Z} \ * \ \mathbb{Z}/2\mathbb{Z} * \ \ldots \ * \ \mathbb{Z}/2\mathbb{Z},$$

the free product of d copies of $\mathbb{Z}/2\mathbb{Z}$, is a d-regular tree, and we can determine that $I_G = d - 2$. This can be verified by considering the effect of adding a single neighboring vertex v to a connected set K. One element (v) is added to K, the edge connecting v to some element in K is no longer a member of ∂K, while the remaining $d-1$ edges connected to v now are. Since a single vertex has d neighbors, we have that

$$|\partial K| = (d-2)|K| + 2$$

This is true independent of K and v, so as we consider the limit, we see $I_G = d-2$.

Using (1.4), we also can compute

$$\beta(T_d) = 1 \qquad \text{and} \qquad \delta(T_d) = \frac{1}{d-1}.$$

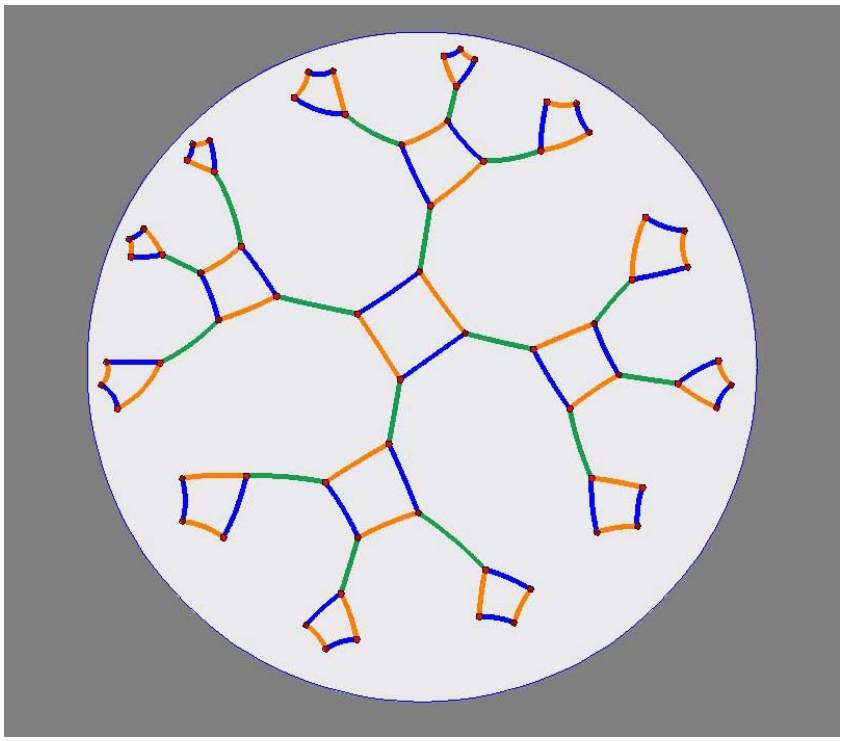

Figure 1.1: The Cayley graph of W_2

1.2.2 One Relator Rank Three Coxeter Groups

Second, consider the groups

$$G = W_m = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^m = 1 \rangle.$$

The Cayley graph of such a group resembles a $2m$-tree where each vertex is replaced by a $2m$-gon. One way to see this is to observe that W_m is a free product of a group of order 4 with $\langle c \rangle$. Figure 1.1 contains a portion of the Cayley graph of W_2.

To calculate the isoperimetric constant I_G, begin by observing that adding one neighboring vertex v to any connected set K adds precisely one edge to ∂K unless v completes a $2m$-gon, in which case $|\partial(K \cup \{v\})| = |\partial K| - 1$. Thus we see that for a fixed size $|K|$, choosing K to include a maximal number of $2m$-gons minimizes $|\partial K|$. Thus to approach I_G we examine a sequence of these sets. Consider the

following sequence of sets. Let K_0 be a $2m$-gon. Recursively define K_i to be the set K_{i-1} unioned with any neighboring "lollypop", i.e. a single neighboring vertex v together with the remainder of the vertices on the $2m$-gon that v is a portion of. This increases the size of K_{i-1} by $2m$, while increasing the size of ∂K_{i-1} by $2m-2$ ($2m-1$ new boundary edges less 1 for the edge connecting v to K_{i-1}, as it is now in the interior). Thus we see that

$$\frac{|\partial K_i|}{|K_i|} = \frac{(2m-2)i + 2m}{2mi + 2m}$$

and in the limit as $i \to \infty$, we see that

$$I_G = \frac{2m-2}{2m}.$$

Employing (1.4), we have

$$\beta(G) = \frac{2m}{2m+1} \qquad \text{and} \qquad \delta(G) = \frac{2m}{4m-1}.$$

1.2.3 One Relator Coxeter Groups of Arbitrary Rank

We now examine a more general case. Consider instead the groups

$$G = W_m^{(d)} = \langle a_1, a_2, \ldots a_d \mid a_1^2 = a_2^2 = \ldots a_d^2 = (a_1 a_2)^m = 1 \rangle.$$

The same process applies as above for identical reasoning. We add $2m$ vertices to K_i in each step, but the boundary grows instead by $2m(d-2)-2$ edges. These are all edges beginning at a vertex on the new $2m$-gon except those that make up the $2m$-gon itself, and we again subtract the single edge making up the handle of the "lollypop". Thus in this instance we have

$$I_G = \frac{2m(d-2)-2}{2m}.$$

Observe further that

$$\lim_{m\to\infty} I_{W_m^d} \to I_{T_d}.$$

This shows that isoperimetric constants behave nicely with these groups, as the Cayley graphs of the groups W_m^d look increasingly like that of T_d as m increases.

Utilizing (1.4), we find that

$$\beta(G) = \frac{2m}{2m+1} \qquad \text{and} \qquad \delta(G) = \frac{2m}{2m(d-1)-1}.$$

1.2.4 Two Relator Rank Three Coxeter Groups

We now examine the groups

$$G = W_{m,n} = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^m = (ac)^n = 1 \rangle.$$

The Cayley graph of G is made up of $2m$-gons and $2n$-gons in such a way that every $2m$-gon has a distinct $2n$-gon attached to every other edge, and likewise for $2n$-gons. However, there are no cycles of polygons. Figures 1.2, 1.3 and 1.4 contain example Cayley graphs (or portions of them). Figure 1.2 was found at [4].

Using identical reasoning to that above, we see that because $n > m$, maximizing the number of $2m$-gons will maximize the number of instances where we reduce the boundary by one. Thus I_G can be approached by the sequence of sets beginning with K_0 a single $2m$-gon and K_i created from K_{i-1} unioned with a neighboring $2n$-gon and all the $2m$-gons adjacent to it. The number of vertices added by this process is $2m(n-1)$, as each of the new vertices is a member of one of the new $2m$-gons. The number of boundary edges added is $(2m-2)(n-1)-2$, as each of the new $2m$-gons contributes $2m-2$ boundary edges to ∂K_i, and we subtract the two edges of the $2n$-gon that were members of ∂K_{i-1}. Thus

$$I_G = \frac{(2m-2)(n-1)-2}{2m(n-1)}$$

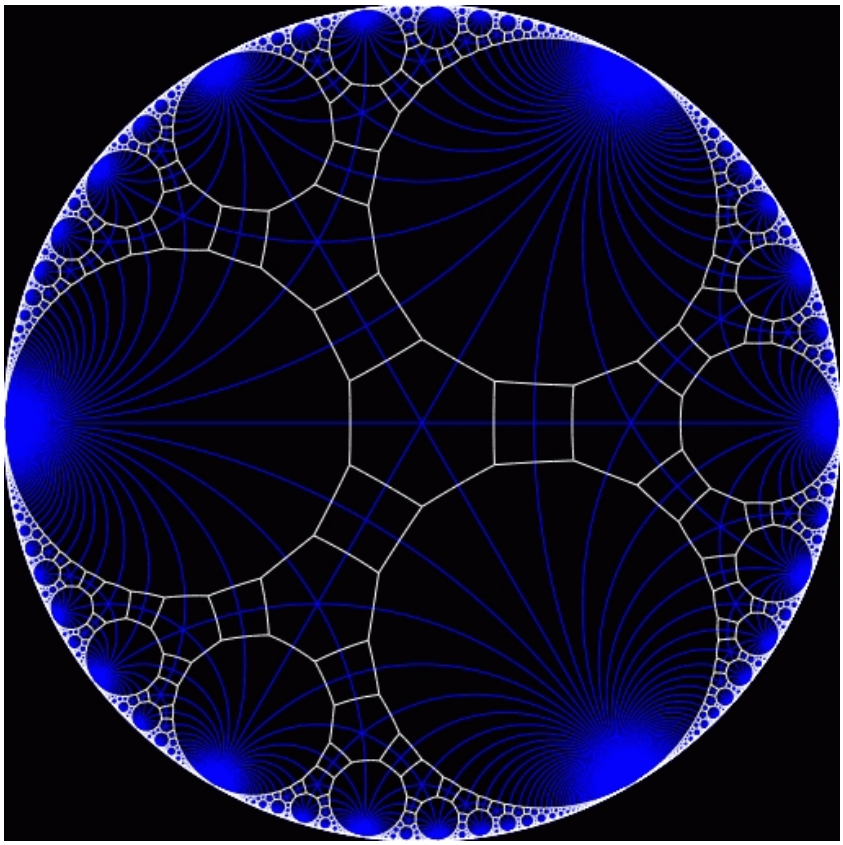

Figure 1.2: The Cayley graph of $W_{2,3}$

Notice that in the limit as $n \to \infty$, we see that $I_{W_{m,n}} \to I_{W_m}$, showing as above that limits of isoperimetric constants in this environment converge as we might hope.

Further calculations with (1.4) show

$$\beta(G) = \frac{2m(n-1)}{(2m+1)(n-1)+1} \qquad \text{and} \qquad \delta(G) = \frac{2m(n-1)}{(4m-1)(n-1)-1}.$$

1.2.5 Two relator Coxeter Groups of Arbitrary Rank

If we now consider the more general class of groups

$$G = \langle a_0, \ldots a_{d-1} \mid a_0^2 = \cdots = a_{d-1}^2 = (a_0 a_1)^{m_1} = \cdots = (a_0 a_{d-1})^{m_{d-1}} \rangle$$

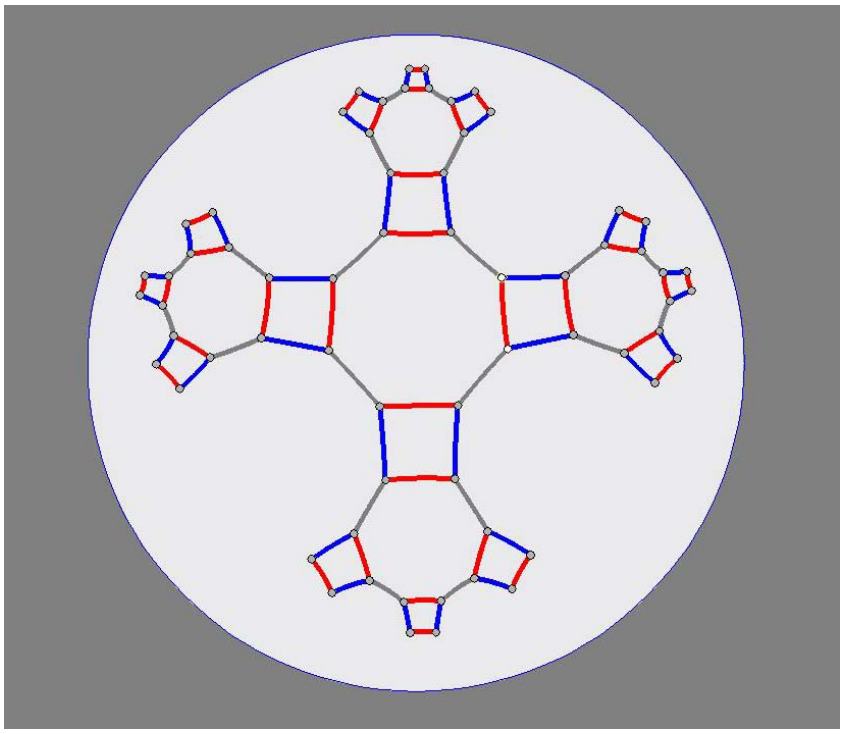

Figure 1.3: The Cayley graph of $W_{2,4}$

with $m_1 \leq m_2 \leq \cdots \leq m_{d-1}$, and m_3 to m_{d-1} allowed to be infinite, we will achieve I_G by maximizing the $2m_1$-gons, and using $2m_2$-gons whenever necessary. Calculating as before, adding one $2m_2$-gon and $m_2 - 1$ $2m_1$-gons increases the number of vertices by $2m_1(m_2 - 1)$ as each new vertex is a member of one of the $2m_1$-gons. The change in the number of edges in the boundary is $(m_2 - 1)\,(2m_1(d-2) - 2) - 2$, as each new vertex adds $d - 2$ edges, except the two on the $2m_2$-gon, which each add one less edge. Again, we subtract the two edges of the $2m_2$-gon which were previously part of the boundary. Thus we see that

$$I_G = \frac{(m_2 - 1)\big(2m_1(d-2) - 2\big) - 2}{2m_1(m_2 - 1)}.$$

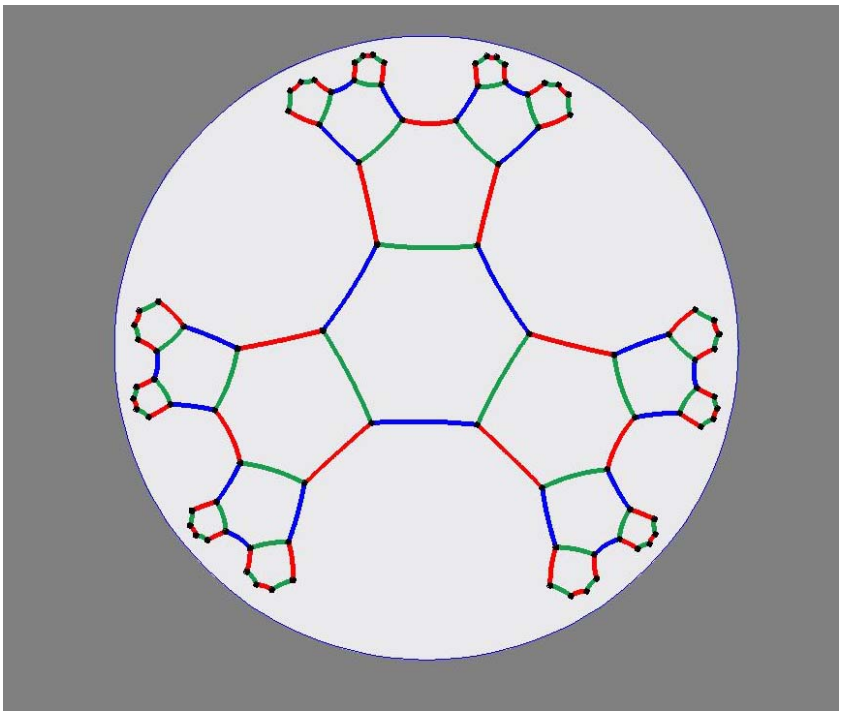

Figure 1.4: The Cayley graph of $W_{3,3}$

1.3 A Lower Bound for Three Relator Rank Three Coxeter Groups

This section is following the methods of Häggström, Jonasson, and Lyons in [3]. We begin by defining a graph to be **semi-regular** if there exist positive integers $m \leq p$ such that the degree of every vertex lies in the interval $[m, p]$. Consider the dual of a planar graph G, denoted by $G_\dagger = (V_\dagger, E_\dagger)$. This graph is semi-regular when G is the Cayley graph of the hyperbolic groups $W_{m,n,p}$, where

$$W_{m,n,p} = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^m = (ac)^n = (bc)^p = 1 \rangle$$

and m, n, and p satisfy

$$2 \leq m \leq n \leq p < \infty \qquad \text{and} \qquad \frac{1}{m} + \frac{1}{n} + \frac{1}{p} < 1.$$

This follows because $G_\dagger$ is a tessellation of the hyperbolic plane by triangles with angles $\frac{\pi}{m}$, $\frac{\pi}{n}$ and $\frac{\pi}{p}$. Note this is also true when any of the indices are infinite, where by convention the corresponding angle(s) of the triangle is 0. The tessellation is obtained by beginning with a single triangle, then adding all its images under reflections in the reflection group generated by the 3 lines in which the triangle's sides lie. Note that in this tessellation all vertices will be of degree $2m$, $2n$, or $2p$. For pictorial examples of these tessellations, some of the figures in Appendix A display both the Cayley graph and dual tessellation.

We will need the following inequality about the graph $G_\dagger$.

Lemma 1.3.1. *If $G_\dagger$ is a semiregular graph, $K \subset V_\dagger$ is any nonempty set and $d_\dagger$ is the minimum degree of the vertices in $G_\dagger$, then* $\dfrac{|K|}{|E^*(K)|} \leq \dfrac{2}{d_\dagger}$,

Proof:

Consider any nonempty set $K \subset V_\dagger$. Note that $E^*(K) = E(K) \cup \partial K$. Combined with the fact that $\frac{|\partial K|}{|K|}$ is nonnegative, this gives

$$\frac{|K|}{|E^*(K)|} = \frac{2|K|}{2|E^*(K)|} = \frac{2}{\frac{|E^*(K)|+|E(K)|}{|K|} + \frac{|\partial K|}{|K|}} \leq \frac{2}{\frac{|E^*(K)|+|E(K)|}{|K|}}$$

The denominator on the right is the "average degree" of K, which must be at least $d_\dagger$, giving the result. □

This lemma implies that

$$\delta(G_\dagger) \leq \frac{2}{d_\dagger}. \tag{1.5}$$

We will also require the following lemma.

Lemma 1.3.2. *For a planar regular graph G with semiregular dual $G_\dagger$, we have $\beta(G) + \delta(G_\dagger) \geq 1$.*

Proof:

Begin by observing that I_G is unchanged if we require K to be connected and

simply connected (when K is viewed as a union of closed faces of $G_\dagger$), as filling in holes increases $|K|$ and decreases $|\partial_E K|$. Since G is regular, (1.4) implies we can also restrict to simply connected sets in calculating $\beta(G)$.

Choose $K_i \subset V$ connected and simply connected (as above) such that $\frac{|K_i|}{|E(K_i)|}$ converges to $\beta(G)$. Let K_i^f be the set of vertices in $G_\dagger$ corresponding to the (bounded) faces of $G(K_i)$. Since the number of faces of the graph $G(K_i)$ is precisely $|K_i^f| + 1$ (by counting the face at ∞), Euler's formula applied to the graph $G(K_i)$ gives the following.

$$|K_i| - |E(K_i)| + |K_i^f| + 1 = 2$$

Rearranging and using $|E^*(K_i^f)| \leq |E(K_i)|$ we have that

$$\begin{aligned} \frac{|K_i|}{|E(K_i)|} + \frac{|K_i^f|}{|E^*(K_i^f)|} &\geq \frac{|K_i|}{|E(K_i)|} + \frac{|K_i^f|}{|E(K_i)|} \\ &= 1 + \frac{1}{|E(K_i)|} \geq 1 \end{aligned}$$

This gives us the lemma as follows:

$$\begin{aligned} \beta(G) + \delta(G_\dagger) &\geq \lim_{i\to\infty} \frac{|K_i|}{|E(K_i)|} + \limsup_{i\to\infty} \frac{|K_i^f|}{|E^*(K_i^f)|} \\ &= \limsup_{i\to\infty} \frac{|K_i|}{|E(K_i)|} + \limsup_{i\to\infty} \frac{|K_i^f|}{|E^*(K_i^f)|} \\ &\geq \limsup_{i\to\infty} \left[\frac{|K_i|}{|E(K_i)|} + \frac{|K_i^f|}{|E^*(K_i^f)|} \right] \\ &\geq 1 \end{aligned}$$

□

Lemma 1.3.2 combined with (1.4) and (1.5) provide a lower bound for I_G.

Lemma 1.3.3. *$I_G \geq (d-2) - \frac{4}{d_\dagger - 2}$, where d is the degree of G and $d_\dagger$ is the minimum degree of $G_\dagger$, as above.*

Chapter 2

Estimates for Connective Constants

2.1 Introduction

The focus of research on self-avoiding walks (SAWs) and self-avoiding polygons (SAPs) has traditionally been centered around the lattices $\mathbb{Z}^d$. More recently, some research has shifted to examine other graphs. Madras and Wu [7] studied regular tilings of the hyperbolic plane, many of which correspond to Coxeter groups. This section focuses on a much larger class of Cayley graphs of Coxeter groups, all of which are hyperbolic, and nearly all are planar.

The primary tool for examining SAWs and SAPs is the connective constant. We begin with a group G and a set S of generators of G. A length k **walk** w on the group G is a sequence of vertices $(w_0, w_1, \ldots, w_k)$ in the Cayley graph of G where w_i and w_{i+1} are neighbors for $i = 0, 1, \ldots, k-1$. We say w is a **SAW** if the $k+1$ sites of w are distinct. A length k **polygon** p is a length k walk w with $w_0 = w_k$. We say p is a **SAP** if the k sites are distinct. We also have a sequence $g_k \in S$ of generators of G for a walk (or polygon) w. These satisfy the property that g_k is the edge label on the Cayley graph of G between the vertices w_{k-1} and w_k.

We denote by c_k (resp. p_k) the number of SAWs (resp. SAPs) of length k. The **connective constants** for SAWs and SAPs are respectively

$$\mu_{w,G} = \limsup_{k\to\infty} (c_k)^{\frac{1}{k}} \qquad \text{and} \qquad \mu_{p,G} = \limsup_{k\to\infty} (p_k)^{\frac{1}{k}} \tag{2.1}$$

We do not specify the choice of generating set, as it will be clear in all uses below. However, different generating sets may give different connective constants (consider

$\mathbb{Z} = \langle 1 \rangle$ versus $\mathbb{Z} = \langle 2, 3 \rangle$). When the context is clear, we additionally will not specify the group.

We now employ the well-known **subadditivity lemma**, which states that for a sequence a_n, if $a_m + a_n \geq a_{m+n}$ for all $m, n \geq 1$ (i.e. a_n is subadditive), then $\lim_{n\to\infty} \frac{a_n}{n}$ exists and equals $\inf_{n\geq 1} \frac{a_n}{n}$. Because c_k is a submultiplicative sequence (i.e. $c_k c_l \geq c_{k+l}$ as any SAW can be broken into two shorter SAWs), using logarithms and the subadditivity lemma we can simplify the connective constant $\mu_{w,G}$ to

$$\mu_{w,G} = \lim_{k\to\infty} (c_k)^{\frac{1}{k}} = \inf_{k\geq 1} (c_k)^{\frac{1}{k}}$$

In this chapter we focus on SAWs, first examining the difference between μ_{w,F_n} and $\mu_{w,G}$, where G is a nontrivial quotient of the free group F_n. We also examine the situation where instead of F_n, we have the "free" Coxeter group T_n. The final result of the chapter is a lower bound on μ_w for a particular class of Coxeter groups.

2.2 Quotients of Free Groups

Consider the free group

$$F_n = \langle a_1, a_2, \ldots, a_n \rangle.$$

Recall that c_{k,F_n} represents the number of self-avoiding walks (SAWs) of length k in F_n. Then

$$c_{k,F_n} = 2n(2n-1)^{k-1},$$

as at any step any generator or its inverse is acceptable except the most recent. This implies that $\mu_{w,F_n} = 2n - 1$.

Lemma 2.2.1. *For any proper quotient G of the free group F_n, we have that*

$$\mu_{w,G} < 2n - 1 = \mu_{w,F_n}.$$

Proof:

Observe first that $\mu_{w,K} \leq \mu_{w,G}$ for any quotient K of G, as every SAW in K lifts to a SAW in G, so $c_{k,K} \leq c_{k,G}$.

Let

$$G_i = \langle a_1, a_2, \ldots a_n \mid r_1, r_2, \ldots, r_i \rangle.$$

Because $\mu_{w,G_i} \leq \mu_{w,G_1} \leq \mu_{w,F_n}$, without loss of generality we need only show the result for one-relator groups G_1. Let $G = G_1$ be such a group.

We obtain an upper bound for the SAWs in G as follows. Let l be the length of the relator r of G. Consider any SAW w of length $k \geq 1$. Because we know of at least one word of length l that cannot be self-avoiding in G, namely r, there are at most $(2n-1)^l - 1$ ways to add l vertices to the end of w to obtain an SAW of length $l + k$. Additionally, we know that there are at most $c_{k,F_n} = 2n(2n-1)^{k-1}$ SAWs of length k. Because c_k is submultiplicative, we have that for any positive integer k,

$$c_k \leq 2n \left((2n-1)^l - 1\right)^{\lfloor \frac{k-1}{l} \rfloor} (2n-1)^{l-1}.$$

This inequality is obtained by considering any first step, then extending it by l segments repeatedly, and then adding at most $l - 1$ steps to achieve the length k.

We now have that

$$\begin{aligned}
\mu_{w,G} &= \lim_{k\to\infty} c_{k,G}^{\frac{1}{k}} \\
&\leq \lim_{k\to\infty} \left(2n\Big((2n-1)^l - 1\Big)^{\lfloor \frac{k}{l} \rfloor}(2n-1)^{l-1}\right)^{\frac{1}{k}} \\
&\leq \left((2n-1)^l - 1\right)^{\frac{1}{l}} \\
&< 2n - 1 = \mu_{w,F_n}
\end{aligned}$$

This proves the result. □

2.3 Quotients of "Free" Coxeter Groups

The following lemma shows that similar conclusions hold with quotients of Coxeter groups.

Lemma 2.3.1. *For any nontrivial quotient W of the group T_d as defined in Section 1.2.1, we have that*

$$\mu_{w,W} < d - 1 = \mu_{w,T_d}.$$

Proof:

Observe first that the number of SAWs in T_d of length k is $d(d-1)^{k-1}$, which shows that $\mu_{w,T_d} = d - 1$.

The remainder of the proof is identical to that of Lemma 2.2.1 by simply replacing every instance of $2n$ with d. □

2.4 A Lower Bound for Three Generator Coxeter Groups

Proposition 2.4.1. *For self-avoiding walks on*

$$W = W_{m,n,p} = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^m = (ac)^n = (bc)^p = 1 \rangle,$$

with $3 \leq m \leq n \leq p$, we have $\mu_w \geq 2^{\frac{2m-5}{2m-4}}$.

Proof:

The proof follows that of Proposition 2.1 in [7].

Partition the sites of the Cayley graph of W into layers, obtained by choosing a single polygon as the first layer, then selecting all neighboring polygons (across an edge) for the second layer, then each layer is determined by all polygons neighboring the previous layer which are not a member of a previous layer. Figure 2.1 contains an example of the first few layers.

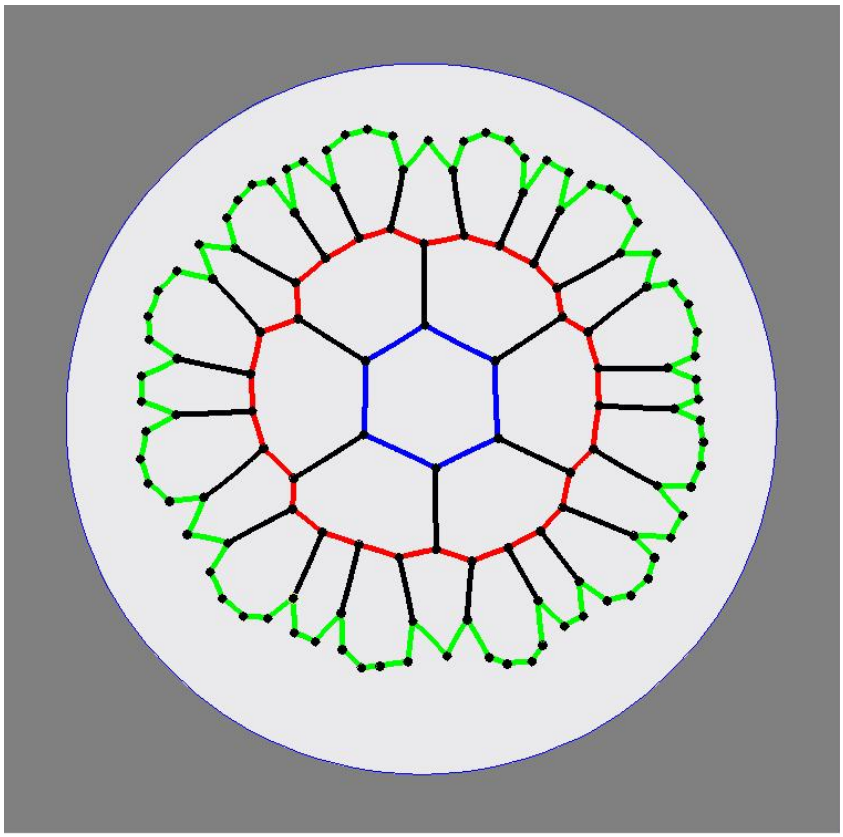

Figure 2.1: The first 3 layers of $W_{3,4,4}$

When $m \geq 3$, each site on any layer but the first has either zero or one neighbor in the previous layer. Define a walk as follows: Begin at any site in the first layer. For the first step, move to the next layer (there is only one choice). Starting now, and any time the walk reaches a new layer, denote by $x+i$ (respectively $x-i$) the ith site from x along the same layer of x in the counterclockwise (resp. clockwise) direction. Notice that each time the walk reaches a site x on a new layer, for $i = 1, 2, \ldots 2m-5$, $x+i$ and $x-i$ each have no neighbor in the previous layer and 1 neighbor in the next layer.

Restrict the walk as follows: Each time a walk reaches a site x on a new layer, it may move to either $x+1$ or $x-1$. If the walk reaches $x+1$ (resp. $x-1$), it has

two choices to move: to $x+2$ (resp. $x-2$) in the same layer or to the next layer. If the walk reaches $x+2m-5$, (resp $x-2m+5$), at the next step we require the walk to move to the next layer. The resulting walk is self-avoiding, as the layers are disjoint. It has the property that if one step of the walk was forced, then it had two choices at the previous $2m-5$ steps. Thus $c_{(2m-4)k} \geq (2^{2m-5})^k$, which implies the proposition. $\square$

Chapter 3

Comparing Self-avoiding Walks and Polygons

3.1 Introduction

In this chapter we examine SAWs and SAPs as defined in Section 2.1. We show that there are exponentially fewer SAPs than SAWs by comparing the connective constants μ_p and μ_w, defined in (2.1). The results below prove $\mu_p < \mu_w$ for most three generator hyperbolic Coxeter groups $W_{m,n,p}$ with $2 \leq m \leq n \leq p \leq \infty$.

3.2 Two or Fewer Relators

3.2.1 A Symmetric Cayley Graph

Assuming $n > 2$, we consider the class of Coxeter groups

$$W_{n,n} = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^n = (ac)^n = 1 \rangle.$$

For reference, a portion of the Cayley graph in the case $n = 3$ is shown in Figure 1.4.

Observe that by reversing the solution to the word problem for Coxeter groups, an SAP in $W_{n,n}$ consists of a collection of connected $2n$-gons. Each SAP can be constructed by beginning with a single $2n$-gon and adding one connected $2n$-gon at a time. Each additional $2n$-gon adds $2n-2$ to the length of the SAP. Thus the length of an SAP made up of k $2n$-gons is $(2n-2)k+2$.

Define a **hinge** h of an SAP p to be 3 consecutive edges in p with generator labels *bab* or *cac*. We say h corresponds to the generator g_k representing the edge

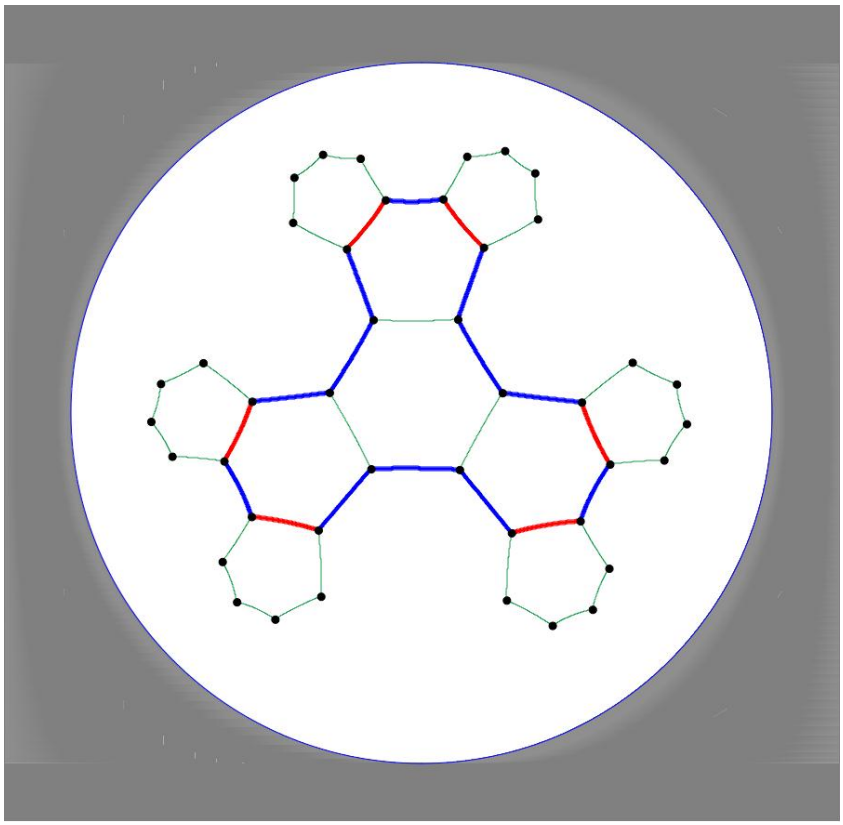

Figure 3.1: Hinges on a SAP in $W_{3,3}$

labeled a. Two hinges h_1 and h_2 of p, corresponding to g_i and g_j respectively, are distinct if $i \neq j$. Notice that, for an SAP, every instance where $g_k = a$ will be a hinge, unless either $k = 1$ or $k = l$.

Intuitively, a hinge is "any (closed) edge in the Cayley graph which, if removed, disconnects the graph". In Figure 3.1, the edges in red are all the hinges of that particular SAP.

Now given a hinge h corresponding to g_k, define a **hinge reflection** to be the walk defined as having vertices w_i, with $w_i = p_i$ for $0 \leq i \leq k$, and using the

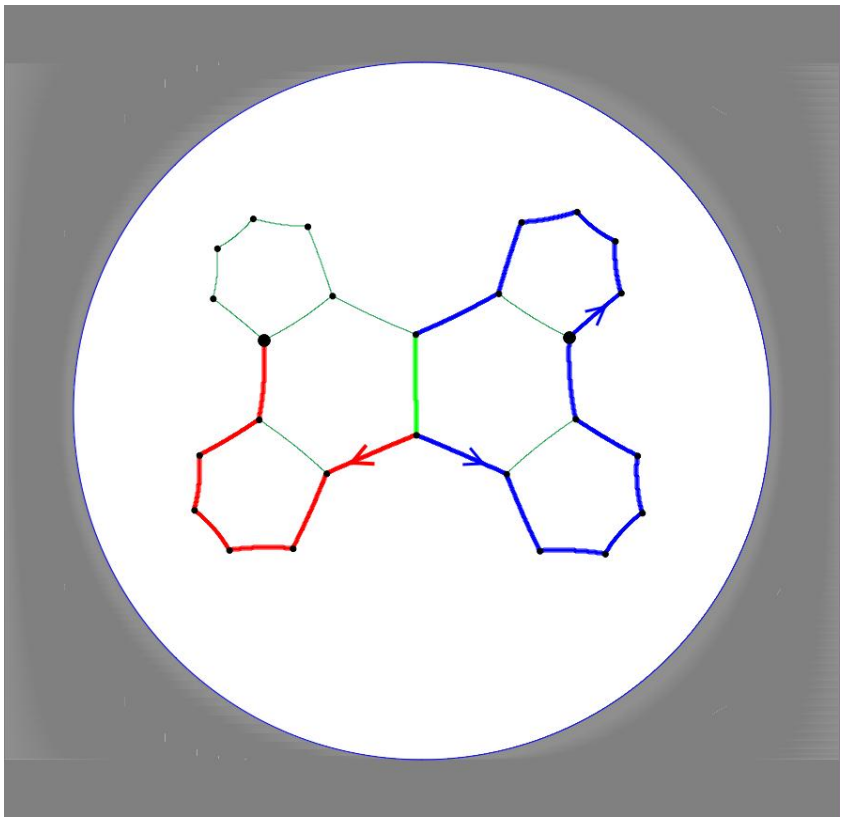

Figure 3.2: A hinge reflection in $W_{3,3}$

following rule for $k < i \leq l$:

$$w_i = \begin{cases} w_{i-1}a & \text{if } p_i = a \\ w_{i-1}b & \text{if } p_i = c \\ w_{i-1}c & \text{if } p_i = b \end{cases} \tag{3.1}$$

If we consider the edge in the Cayley graph corresponding to g_k, let L_k be the line in $\mathbb{H}^2$ that this edge is a segment of. The hinge reflection h is the reflection of the portion of the SAP following h across the line L_k. Note that the vertices w_k and w_{k-1} are stable under this reflection. In Figure 3.2, the edges in red are the result of the hinge reflection corresponding to the green edge.

Lemma 3.2.1. *The walk w resulting from a hinge reflection is self-avoiding.*

Proof:

Assume not. Then we must have $w_i = w_j$ for $i < j$. Without loss of generality we may assume that the portion of w between w_i and w_j is self-avoiding. Consider the following cases:

$j \leq k$:

This would imply that we did not begin with an SAP, a contradiction.

$k - 1 \leq i$:

Notice that the isomorphism of $W_{n,n}$ defined by sending the generators $a,\ b$, and c to $a,\ c$, and b respectively maps SAPs to SAPs and SAWs to SAWs. This case also, therefore, would imply that we did not begin with an SAP, again a contradiction.

$i < k - 1$, $j > k$:

We assumed that the segment between w_i and w_j is self-avoiding, and that $w_i = w_j$. This implies that this segment defines an SAP. But the definition of a hinge reflection implies that this SAP must include the sequence *cab* or *bac* representing the edges from w_{k-2} to $w_k + 1$.

However, no SAP can contain this sequence, as verified by examining the construction of any SAP from $2n$-gons. Since the only relators in the group are $(ab)^n$ and $(ac)^n$, we can build an SAP by beginning with a single $2n$-gon and adding others. In each step of this process, a section *bab* may be replaced by $b(ca)^{n-1}cb$, or a section *cac* may be replaced by $c(ba)^{n-1}bc$. Neither of these steps can introduce the sequences *bac* or *cab*, and therefore no polygon may contain either of the sequences.

Therefore it cannot be that $w_i = w_j$ for $i < j$, verifying that w is an SAW. □

An important observation is that the proof only required the initial sequence to be self-avoiding, not necessarily a polygon. This gives the following corollary:

Lemma 3.2.2. *Performing a hinge reflection on an SAP or SAW yields an SAW.*

Another observation is that, after performing a hinge reflection over h in an SAP or SAW, all hinges except h in the original walk or polygon persist as hinges in the resulting SAW (although possibly reflected to a different location). Therefore, Lemma 3.2.2 implies that we can perform multiple hinge reflections on an SAP and the result will be a SAW.

3.2.2 Groups With Less Symmetry

For $2 \leq m < n < \infty$ we consider the groups

$$W_{m,n} = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^m = (ac)^n = 1 \rangle.$$

For these groups the same definition of hinge applies, but the definition of hinge reflection is no longer valid, as $W_{m,n}$ is not symmetric with respect to b and c. For examples, see Figures 1.2 and 1.3. Begin with an SAP P in $W_{m,n}$. Recall the notation of p_i for vertices of P and g_i for the generator of $W_{m,n}$ corresponding to the segment from p_{i-1} to p_i. Let h be a hinge in P corresponding to the edge p_{k-1} to p_k (which implies $g_k = a$).

We define a **hinge reflection** of P over h to be the SAW defined by:

$$w_i = \begin{cases} p_i & \text{if } 0 \leq i \leq k \\ w_k b & \text{if } i = k+1 \text{ and } g_{k+1} = c \\ w_k c & \text{if } i = k+1 \text{ and } g_{k+1} = b \\ w_{i-1} g_{i-1} & \text{if } i > k+1 \end{cases}$$

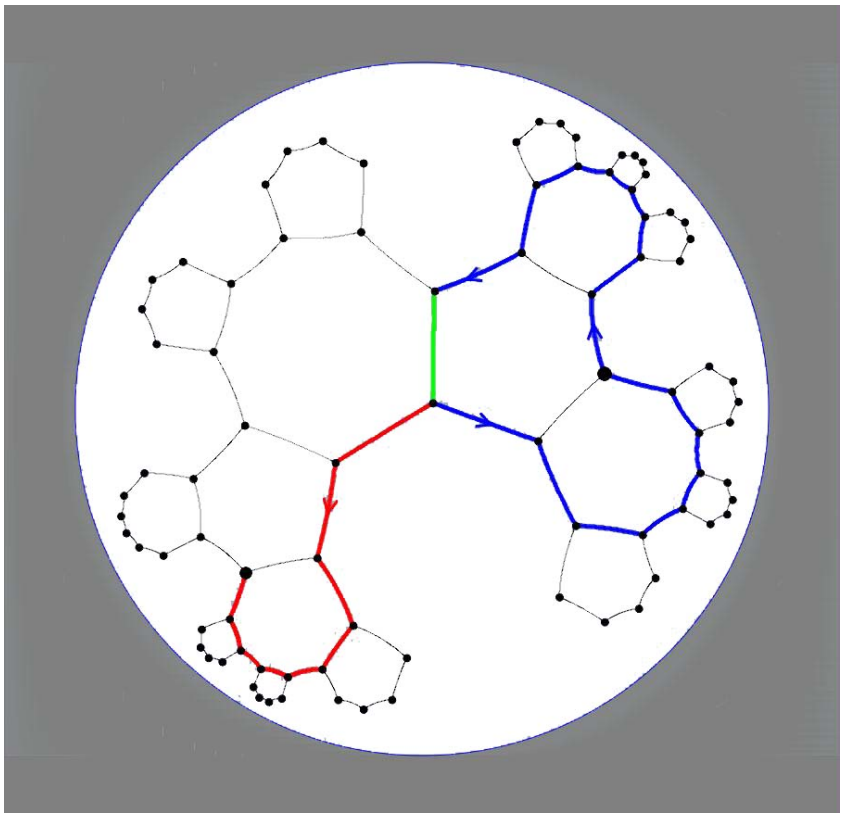

Figure 3.3: A hinge reflection in $W_{3,4}$

This is a reflection of the end of P across the line perpendicular to the edge between w_i and w_{i+1} (which was not originally in P). Note that this SAW is one step longer than the original SAP or SAW. Figure 3.3 is an example of a hinge reflection when $m = 3$ and $n = 4$. The blue edges are the original SAP, the green edge is the hinge, and the red edges are the new tail of the SAP after the hinge reflection.

Both Lemma 3.2.1 and Lemma 3.2.2 hold on groups of the form $W_{m,n}$ for precisely the reason they hold for $W_{n,n}$, as no SAP on either group can contain a sequence of edges corresponding to *bac* or *cab*.

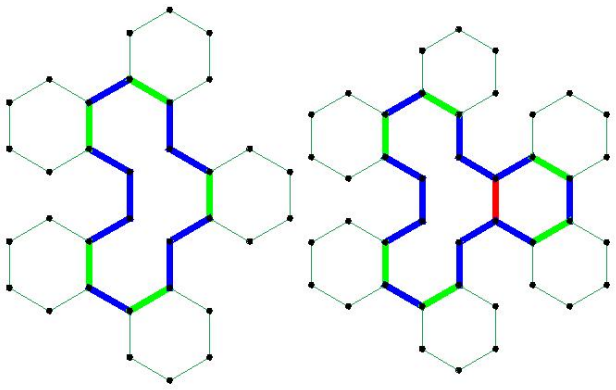

Figure 3.4: Adding one hexagon to a SAP in $W_{3,3}$

3.2.3 Counting Hinges

We now count the number of hinges appearing in a SAP according to its length. Notice that creating an SAP in $W_{3,3}$ consists of adding hexagons, and that adding a single hexagon destroys one hinge, but creates two more, as shown in Figure 3.4.

This gives us the following lemmas:

Lemma 3.2.3. *The number of hinges in any SAP of length k in the group $W_{m,n}$ with $2 < m \leq n < \infty$ is at least*

$$\left\lfloor \frac{k}{2n-2} \right\rfloor + 1$$

Proof:

Observe first that the most a $2m$- or $2n$-gon can add to the length of an SAP is $2n-2$. Therefore there are at least

$$\left\lfloor \frac{k}{2n-2} \right\rfloor$$

*-gons in an SAP of length k. Now note that the initial *-gon has at least $m-1$ hinges (one a may be adjacent to the initial vertex), and any connected *-gons

increase the number of hinges by at least $m-2$. Because $m-2 \geq 1$, any SAP P has at least one more hinge than the number of *-gons that make up P. The lemma follows. □

Lemma 3.2.4. *The number of hinges in an SAP of length k in the group $W_{2,n}$ with $2 < n < \infty$ is at least*

$$(n-2)\left\lceil \frac{k}{4n-4} \right\rceil + 1$$

Proof:

There can be no more than $r(n-1)+1$ squares in an SAP with r $2n$-gons (the first $2n$-gon will have at most n squares adjacent to it, and all other $2n$-gons can have at most $(n-1)$ additional squares adjacent). Therefore because $4n-4$ is the change in length from adding a $2n$-gon and all adjacent squares, the least number of $2n$-gons in an SAP of length k is

$$\left\lceil \frac{k-2}{4n-4} \right\rceil$$

Observing that an initial square has one hinge, and an initial $2n$-gon has $n-1$ hinges, we can add one to our total number of hinges. Each additional $2n$-gon adds $n-2$ hinges. Since $n-2 \geq 1$, the lemma follows. □

3.2.4 Uniqueness

From above, we see that performing a hinge reflection on a SAP in $W_{n,n}$ leaves a "calling card" of a sequence of edges labelled either *bac* or *cab*. Therefore, given any SAW obtained from a hinge reflection (or series of hinge reflections) on an SAP, we can reconstruct the original SAP by stepping through the walk, performing a hinge reflection wherever we find *bac* or *cab*. This implies the following lemma.

Lemma 3.2.5. *Assume P_1 and P_2 are SAPs in $W_{m,n}$ of length r. Then for any SAWs w_1 and w_2 obtained by hinge reflections on P_1 and P_2 respectively, then*

(1) if we can extend w_1 or w_2 to be an arbitrary length SAW with a mild assumption. In particular, we can extend either to have length at least

$$r + \left\lceil \frac{r}{2n-2} \right\rceil .$$

(2) $w_1 = w_2$ if and only if $P_1 = P_2$ and the same set of hinges is used to obtain both w_1 and w_2.

Proof: (1)

Given an SAW w obtained from P by hinge reflections (at least one), we assume that the "last" hinge, call it h_*, in P is used. This means that all the edges following h_* carry the label b or c. We can then extend w by the requisite number of edges by adding edges alternatingly corresponding to b or c, starting with whichever did not come last, until w is sufficiently long. Since the edge h_* divides the Cayley graph in two pieces, and the portion of w following h_* has no edges labeled a, the result is self-avoiding. Note that if the final edge is labeled a, then choose b or c according to whichever is not the label of the penultimate edge.

(2)

First assume $P_1 = P_2$ and H is the set of hinges used to obtain both w_1 and w_2. Then we must show $w_1 = w_2$. This will be the case if hinge reflections commute. Note that when $m = n$ a hinge reflection only exchanges the edge labels b and c following the hinge. Thus if we use two hinge reflections h_i and h_j, between i and j the edge labels will be switched once and after j they will be returned to the original. This is regardless of ordering, so hinge reflections commute. Now let $m \leq n$ and use the second definition of a hinge reflection. Note that in this

case the sole effect of a hinge reflection on the sequence g_i is inserting a term into the sequence immediately after the hinge, incrementing the remaining subscripts. Thus two hinge reflections h_i and h_j with $i < j$ will insert a generator between g_i and g_{i+1} and betweeen what are initially g_j and g_{j+1} in the original polygon or walk, and regardless of ordering. Therefore we see that $w_1 = w_2$, as both satisfy this equation.

Now if $w_1 = w_2$, then by searching w_1 and w_2 for the sequences *bacb* and *cabc*, we can locate where hinge reflections were performed. From these we can reconstruct P_1 and P_2, and since $w_1 = w_2$, they will be equal and the hinge reflections used will be identical. □

3.2.5 There Are Fewer Polygons Than Walks

Theorem 3.2.1. *$\mu_p < \mu_w$ for all Coxeter groups of the form*

$$W = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^m = (ac)^n = 1 \rangle, \text{ with } 2 \leq m \leq n \leq \infty,$$

Proof: We break the proof into cases as follows.

Case 1: $m = n = 2$

Observe that this Cayley graph is an infinite ladder. For any even number $k \geq 4$ there is exactly 1 self-avoiding polygon of length k, up to translation and reflection. Therefore, we see that $p_k = 2k$, and thus

$$\limsup_{k \to \infty} (p_k)^{\frac{1}{k}} = 1 = \mu_p.$$

Zeilberger in [13] proves that

$$\mu_w = \frac{1 + \sqrt{5}}{2},$$

the Golden Ratio. Therefore, the theorem holds for this case.

Case 2: $m = 2,\ 2 < n < \infty$

From Lemma 3.2.4, the assumption in Lemma 3.2.5, and $n \geq 3$ we see that for any SAP of length k there are

$$\left\lceil \frac{k}{4n-4} \right\rceil$$

hinges that are free to use. For convenience, let t be this number. Lemma 3.2.2 shows that any SAW resulting from using any subset of the t (remaining) hinge reflections is an SAW. Thus for every SAP of length k, there are 2^t distinct SAWs associated to it of length $k+t$, and Lemma 3.2.5 implies all the SAWs are distinct. Thus we have

$$c_{k+t} \geq 2^t p_k$$

We now must show that this implies that $\mu_w > \mu_p$. First, we calculate as follows.

$$\begin{aligned} \mu_w &= \lim_{(k+t)\to\infty} \Big(c_{k+t}\Big)^{\frac{1}{k+t}} \\ &\geq \lim_{(k+t)\to\infty} \Big((2^t)p_k\Big)^{\frac{1}{k+t}} \\ &= \lim_{(k+t)\to\infty} (2^t)^{\frac{1}{k+t}} \lim_{(k+t)\to\infty} \Big(p_k\Big)^{\frac{1}{k+t}} \end{aligned}$$

Calculating, we find that

$$\frac{t}{k+t} \geq \frac{\frac{k}{4n-4}}{\frac{4n-3}{4n-4}k+1} = \frac{1}{4n-3+\frac{4n-4}{k}},$$

which implies that

$$\lim_{k\to\infty} 2^{\frac{t}{k+t}} \geq 2^{\frac{1}{4n-3}}.$$

We also have that

$$\frac{1}{k+t} \geq \frac{1}{k+\frac{k}{4n-4}+1} = \frac{4n-4}{(4n-3)k+4n-4}$$

which gives

$$\lim_{(k+t)\to\infty} \Big(p_k\Big)^{\frac{1}{k+t}} = \mu_p^{\frac{(4n-4)}{(4n-3)}}.$$

We can use the fact from Lemma 2.3.1 that $\mu_p \leq \mu_w < 2$, and we have

$$\mu_w \geq 2^{\frac{1}{4n-3}} \mu_p^{\frac{4n-4}{4n-3}} = \left(\frac{2}{\mu_p}\right)^{\frac{1}{4n-3}} \mu_p > \mu_p.$$

Case 3: $2 < m = n < \infty$

Lemma 3.2.3 and the assumption in Lemma 3.2.5 imply that any SAP of length k has at least

$$(n-2)\left\lfloor \frac{k}{2n-2} \right\rfloor$$

hinges that are free to use. Lemma 3.2.2 implies that performing any number of consecutive hinge reflections leaves us with a SAW. Therefore by Lemma 3.2.5, we can choose any combination of the hinges in an SAP to reflect, which implies that for every SAP of length k, there are at least

$$2^{(n-2)\lfloor \frac{k}{2n-2} \rfloor}$$

SAWs of length k associated to it. Therefore, we have that

$$p_k < 2^{\frac{(n-2)k}{2n-2} - (n-2)} p_k \leq c_k$$

Taking the limit, we see that

$$\begin{aligned}
\mu_w &\geq \lim_{k\to\infty} \left(2^{\frac{k(n-2)}{2n-2} - (n-2)} p_k\right)^{\frac{1}{k}} \\
&= \lim_{k\to\infty} \left(2^{\frac{n-2}{2n-2} - \frac{n-2}{k}}\right) \lim_{k\to\infty} p_k^{\frac{1}{k}} \\
&= 2^{\frac{n-2}{2n-2}} \mu_p \\
&> \mu_p
\end{aligned}$$

and therefore the Theorem holds in this case.

Case 4: $2 < m < n < \infty$

Lemma 3.2.3 and the assumption in Lemma 3.2.5 imply that any SAP of length k has at least

$$\left\lfloor \frac{k}{2n-2} \right\rfloor$$

hinges that are free to use. For convenience, let t represent this number. Lemma 3.2.2 implies that performing any number of consecutive hinge reflections leaves us with an SAW. Therefore, we can choose any combination of the hinges in an SAP to reflect, which implies that for every SAP of length k, there are at least 2^t SAWs associated to it. We know both that these SAWs are of length $t + k$ (or can be extended to that length) and that they are all distinct by Lemma 3.2.5. Therefore, we have that

$$c_{k+t} \geq (2^t)p_k$$

We now must show that this implies that $\mu_w > \mu_p$. First, we calculate as follows.

$$\begin{aligned} \mu_w &= \lim_{(k+t)\to\infty} \Big(c_{k+t}\Big)^{\frac{1}{k+t}} \\ &\geq \lim_{(k+t)\to\infty} \Big((2^t)p_k\Big)^{\frac{1}{k+t}} \\ &= \lim_{(k+t)\to\infty} (2^t)^{\frac{1}{k+t}} \lim_{(k+t)\to\infty} \Big(p_k\Big)^{\frac{1}{k+t}} \end{aligned}$$

Because

$$\frac{t}{k+t} \geq \frac{\frac{k}{2n-2} - 1}{\frac{2n-1}{2n-2}k} = \frac{1}{2n-1} - \frac{(2n-1)}{(2n-2)k},$$

we have that

$$\lim_{k\to\infty} 2^{\frac{t}{k+t}} \geq 2^{\frac{1}{2n-1}}.$$

Because we also have that

$$\frac{1}{k+t} \geq \frac{1}{k+\frac{k}{2n-2}} = \frac{2n-2}{(2n-1)k}$$

we can use the fact from Lemma 2.3.1 that $\mu_p \leq \mu_w < 2$, and we end up with

$$\mu_w \geq 2^{\frac{1}{2n-1}} \mu_p^{\frac{2n-2}{2n-1}} = \left(\frac{2}{\mu_p}\right)^{\frac{1}{2n-1}} \mu_p > \mu_p$$

completing the proof in this case.

Case 5: $2 \leq m < \infty,\ n = \infty$

See Figure 1.1 for an example Cayley graph of such a group. There is a single SAP of length $2n$ and none of any greater length, so $\mu_p = 0$. To show that there are many SAWs, consider all walks of the form $c * c * c * c \ldots$, where each asterisk can be replaced by either a or b. These are all SAWs because nowhere does there appear a subword ab, and because there is a choice every 2 steps, there are $2^{\lfloor \frac{k}{2} \rfloor}$ such walks of length k. Therefore $\mu_w \geq \sqrt{2}$, and the theorem holds.

Case 6: $n = m = \infty$

The Cayley graph in this instance is a tree with degree 3, which has no SAPs. Therefore $\mu_p = 0$. Also, there are $3 \cdot 2^{n-1}$ walks of length n, and so $\mu_w = 2$.

Thus we have shown the theorem holds. □

3.3 Three Relator Rank Three Coxeter Groups

Denote by R the spectral radius associated to the adjacency matrix of G, let $K \subset V$ be a subset of sites, let ∂K and I_G be as defined in Section 1.1.

We will need a proposition proved in [7] as Proposition 5.2. It is stated below for completeness:

Proposition 3.3.1. *Let R be the spectral radius of the Cayley graph of $W_{m,n,p}$. Then*

$$\mu_p \leq \frac{R + \sqrt{R^2 - 8}}{2}$$

Theorem 3.3.2. *For all Coxeter groups*

$$W_{m,n,p} = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^m = (ac)^n = (bc)^p = 1 \rangle,$$

with $7 \leq m \leq n \leq p < \infty$, it is the case that $\mu_p < \mu_w$.

Proof:

Begin by observing that each of these groups has a Cayley graph of degree 3. Utilizing Proposition 3.3.1, all that remains to be shown is that $\frac{R+\sqrt{R^2-8}}{2} < \mu_w$. As proved in [9], $I_G^2 + R^2 \leq D^2$, where D is the degree of the graph. Rearranging, we see that $R \leq \sqrt{3^2 - I_G^2}$. Substituting the estimate for I_G in Lemma (1.3.3) into this equation with $d_\dagger = 2m$, we have

$$\begin{aligned}
\mu_p &\leq \frac{R + \sqrt{R^2 - 8}}{2} \\
&\leq \frac{\sqrt{9 - I_G^2} + \sqrt{1 - I_G^2}}{2} \\
&\leq \frac{\sqrt{9 - \left(\frac{m-3}{m-1}\right)^2} + \sqrt{1 - \left(\frac{m-3}{m-1}\right)^2}}{2} \\
&\leq \frac{\sqrt{8m^2 - 12m} + \sqrt{4m - 8}}{2(m-1)} \\
&\leq \frac{\sqrt{2m^2 - 3m} + \sqrt{m - 2}}{m - 1}
\end{aligned}$$

and it is easily verified that when $m \geq 7$ this estimate is strictly less than the estimate for μ_w given in Prop 2.4.1, which gives the result. □

Note that this proof generalizes easily to the following result.

Theorem 3.3.3. *If G is any tiling with degree three and semi-regular dual (defined in 1.3) with minimum degree at least 14, then $\mu_p < \mu_w$.*

3.4 Nontrivial Free Products

Theorem 3.4.1. *For any non-trivial free product $G = G_1 * G_2$, $\mu_{w,G} > \mu_{p,G}$.*

Proof:

Begin with a group G that is a free product of a group G_1 and G_2. Note first that if $|G_1| = |G_2| = 2$, the Cayley graph of G is a line. Thus $\mu_{p,G} = 0$ as there are no polygons of any positive length. Also, $\mu_{w,G} = 1$, as there are precisely 2 SAWs of any positive length.

What remains is to show the result in the case that $|G_1| \geq 3$ and $|G_2| \geq 2$. Observe first that $\mu_{p,G} = \max\{\mu_{p,G_1},\ \mu_{p,G_2}\}$ because there are no additional relators in G beyond those in G_1 and G_2. Without loss of generality we may assume $\mu_{p,G_1} \geq \mu_{p,G_2}$ because if $|G_2| = 2$, then $\mu_{p,G_2} = 0$, and in all other cases we may reorder G_1 and G_2. It is now enough to show that $\mu_{w,G} > \mu_{w,G_1}$.

Begin with the generating function $f(t)$ defined as

$$\sum_{k=0}^{\infty} c_k t^k$$

where c_k is the number of SAWs of length n in G (for more information on generating functions, see [12]). Define $f_1(t)$ similarly, and let g_2 be any nontrivial generator of G_2.

Considering SAWs in G, the following are all SAWs in G: g_2 followed by a nontrivial walk in G_1, repeated any nonnegative number of times. Observe that each repetition of the second step contributes $t\big(f_1(t)-1\big)$ to the generating function.

The generating function is thus the sum of powers of the last, which means we have the generating function

$$\bar{f}(t) = \frac{1}{1 - t\big(f_1(t) - 1\big)}$$

where $\bar{f}(t)$ satisfies the property that the coefficient of its nth term is less than or equal to the coefficient for the nth term of $f(t)$. Now $\mu_{w,G}$ is the multiplicative inverse of the smallest positive pole of $f(t)$, and the relation above implies that $\mu_{w,G} \geq$ the multiplicative inverse of the smallest pole of $\bar{f}(t)$. Looking at all the poles of $\bar{f}(t)$, we find that they consist of the poles of $f_1(t)$ and anywhere the denominator is 0. Thus to calculate any additional poles we need $1+t-tf_1(t) = 0$ or, by rearranging,

$$\frac{1+t}{t} = f_1(t)$$

We know that $\frac{1}{\mu_{w,G_1}}$ is the smallest positive pole of $f_1(t)$, that $f_1(0) = 1$, and also that $f_1(t)$ is a continuous function on the interval $[0, \frac{1}{\mu_{w,G_1}})$. Examining $\frac{1+t}{t}$, we see that at $\frac{1}{\mu_{w,G_1}}$, it has the value $\mu_{w,G_1} + 1$, and also

$$\lim_{t \to 0^+} \frac{1+t}{t} = \infty$$

Therefore these functions intersect in the interval $\left(0, \frac{1}{\mu_{w,G_1}}\right)$. Call this point c. Then c is a pole of $\bar{f}(t)$ and $\mu_{w,G_1} < \frac{1}{c} \leq \mu_{w,G}$. Therefore $\mu_{w,G} > \mu_{w,G_1} \geq \mu_{p,G}$. $\square$

Chapter 4

The Future

In this section we suggest further avenues of research. Considering isoperimetric constants, few results show comparisons of the isoperimetric constants of similar groups. The results in [3], the isoperimetric constants calculated in Chapter 1 and the form of the lower bounds in Lemma 1.3.3 suggest the following conjecture:

Conjecture 4.1. *Using the notation $I_{m,n,p}$ to represent the isoperimetric constant of a hyperbolic group $W_{m,n,p}$, with $m \leq n \leq p < \infty$,*

$$I_{m,n,p} \leq I_{m,n,p+1}, \qquad I_{m,n,p} \leq I_{m,n+1,p} \qquad \textit{and} \qquad I_{m,n,p} \leq I_{m+1,n,p}$$

Note that if one or more of the indices is infinite, Chapter 1 proves those cases.

Further results may also be found in comparing the connective constants of a group G versus one of its subgroups H when G is not free (or "free"). An important note is that the choice of presentation influences the connective constant, so some restrictions may be necessary. As in Chapter 2, strict inequalities between connective constants are the goal, as non-strict inequalities are trivial if the presentations are fixed.

A final avenue of approach relates to the result of Theorem 3.3.2. It remains open for hyperbolic groups $W_{m,n,p}$ with $m \leq 6$, n, p finite. Proceeding well beyond this, our intuition suggests that this result should hold for all infinite hyperbolic groups (Madras and Wu also conjecture this in [7] following Theorem 1.2 — they further suggest this may be true for nonamenable graphs or graphs with exponential growth).

Appendix A

Example Cayley Graphs

To provide some insight into the Cayley graphs that may appear, this appendix contains a variety of images. Most examples are of groups of the type $W_{m,n,p}$. Included at the end are examples of higher degree Coxeter groups. The necessary definitions are

$$\begin{aligned}
W_{2,2,2,3} &= \langle a_1, a_2, a_3, a_4 \ : \ a_i^2, (a_1a_2)^2, (a_2a_3)^2, (a_3a_4)^2, (a_1a_4)^3 \rangle \\
W_{2,3,2,3} &= \langle a_1, a_2, a_3, a_4 \ : \ a_i^2, (a_1a_2)^2, (a_2a_3)^3, (a_3a_4)^2, (a_1a_4)^3 \rangle \\
W_{2,2,3,4} &= \langle a_1, a_2, a_3, a_4 \ : \ a_i^2, (a_1a_2)^2, (a_2a_3)^2, (a_3a_4)^3, (a_1a_4)^4 \rangle \\
W_{2,2,2,2,2} &= \langle a_1, a_2, a_3, a_4, a_5 \mid \\
&\qquad a_i^2, (a_1a_2)^2, (a_2a_3)^2, (a_3a_4)^2, (a_4a_5)^2, (a_1a_5)^2 \rangle
\end{aligned}$$

Note that for nearly all of the examples in this appendix, the question of whether $\mu_w > \mu_p$ remains an open question. All images below were created by [4] and [8].

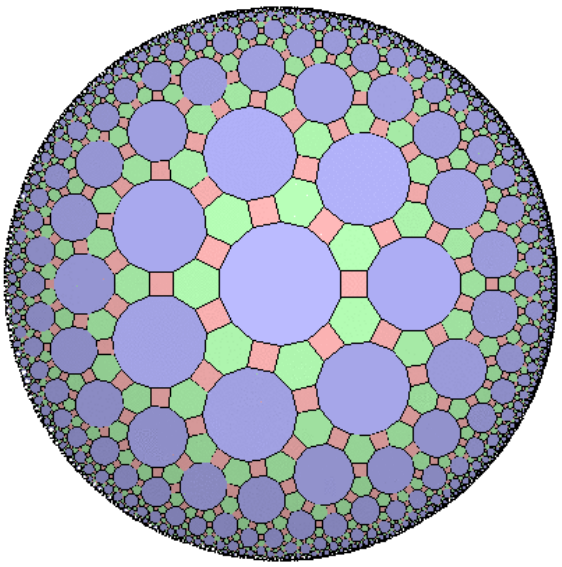

Figure A.1: The Cayley graph of $W_{2,3,7}$

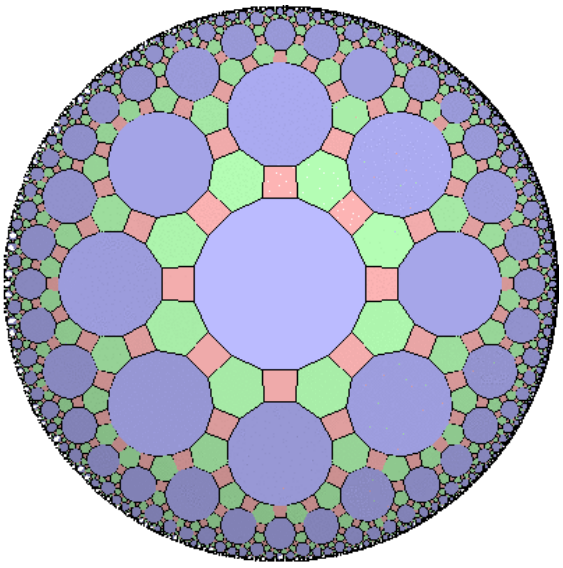

Figure A.2: The Cayley graph of $W_{2,3,8}$

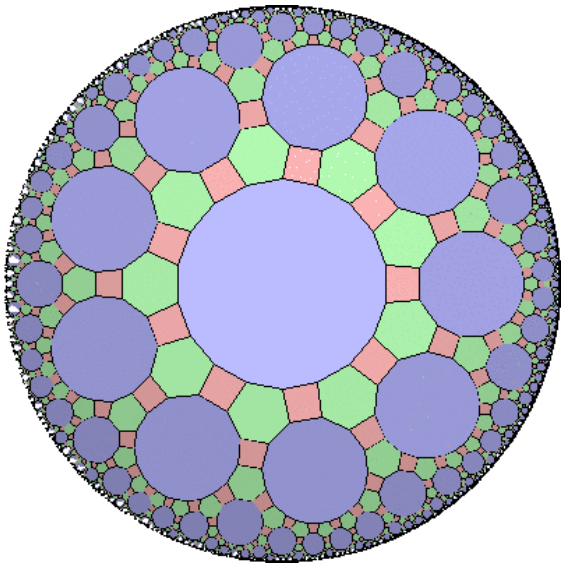

Figure A.3: The Cayley graph of $W_{2,3,9}$

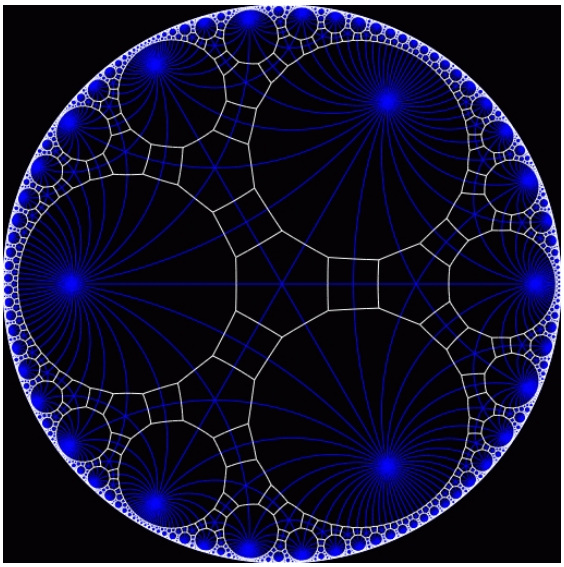

Figure A.4: The Cayley graph of $W_{2,3,20}$

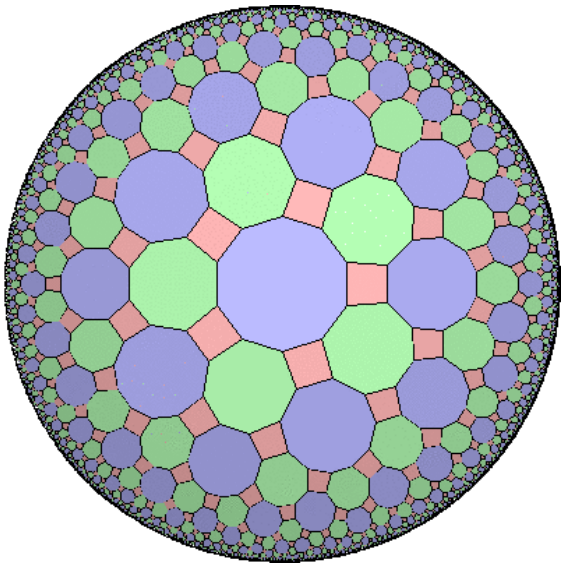

Figure A.5: The Cayley graph of $W_{2,4,5}$

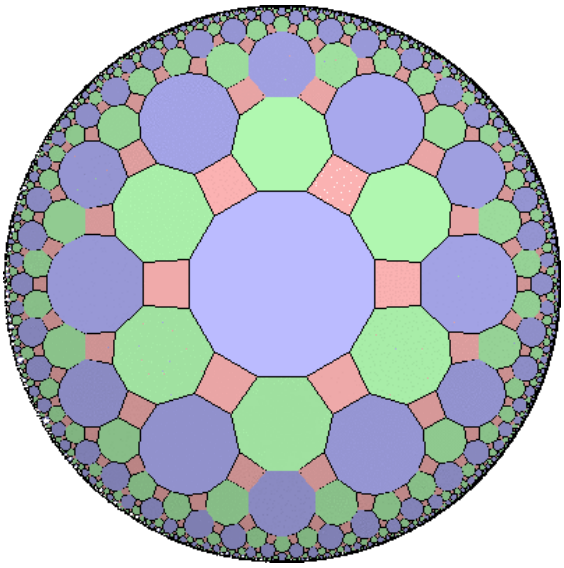

Figure A.6: The Cayley graph of $W_{2,4,6}$

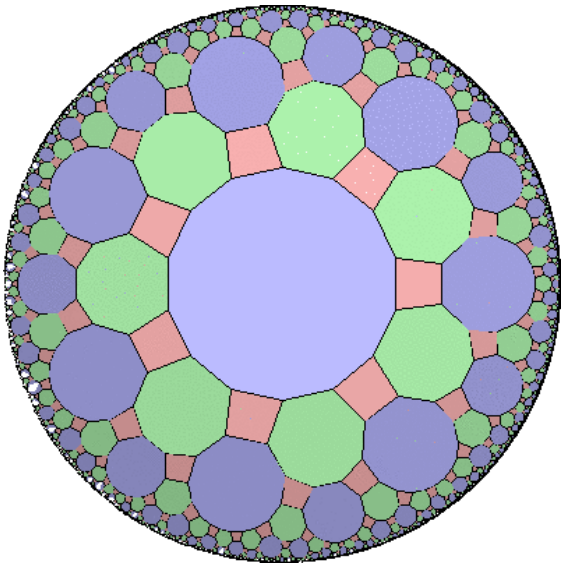

Figure A.7: The Cayley graph of $W_{2,4,7}$

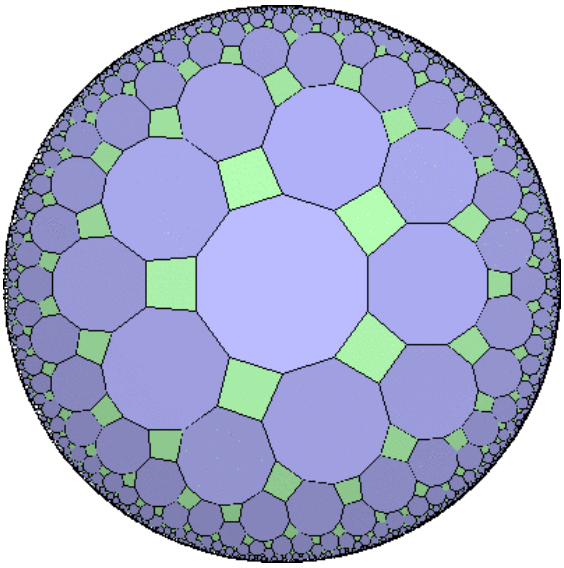

Figure A.8: The Cayley graph of $W_{2,5,5}$

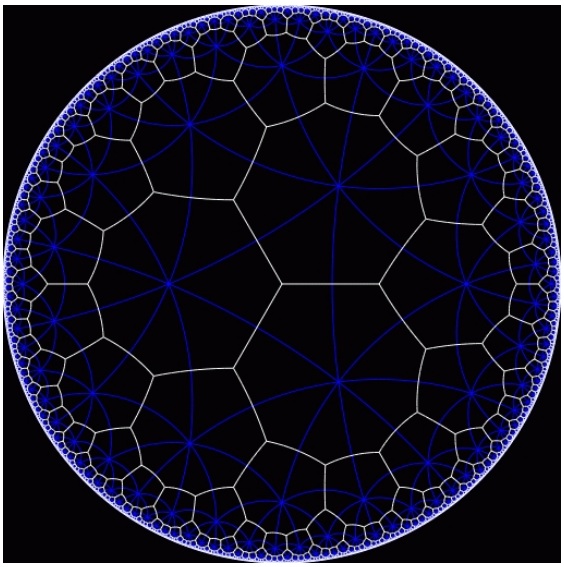

Figure A.9: The Cayley graph of $W_{4,4,4}$

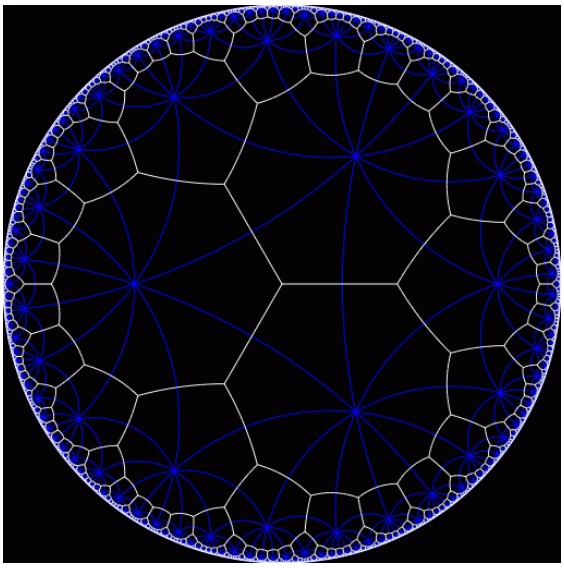

Figure A.10: The Cayley graph of $W_{5,5,5}$

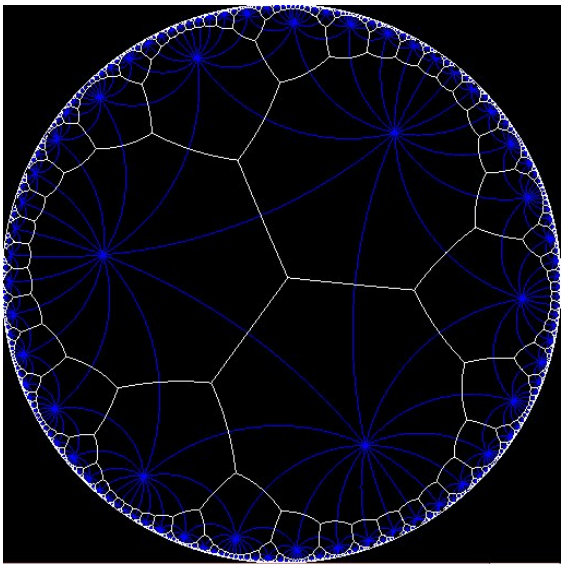

Figure A.11: The Cayley graph of $W_{7,7,7}$

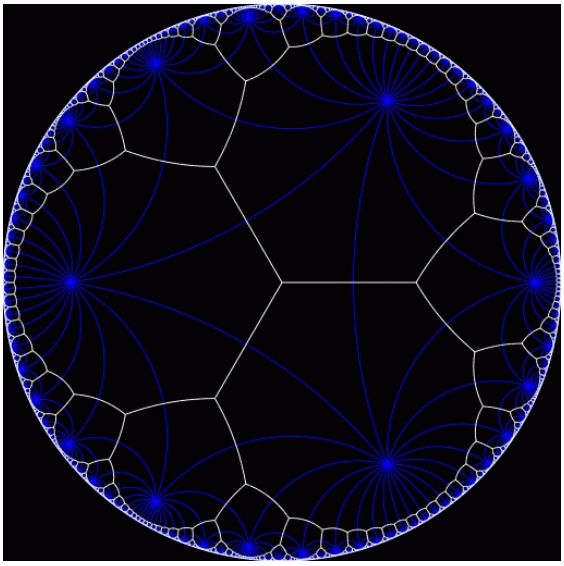

Figure A.12: The Cayley graph of $W_{10,10,10}$

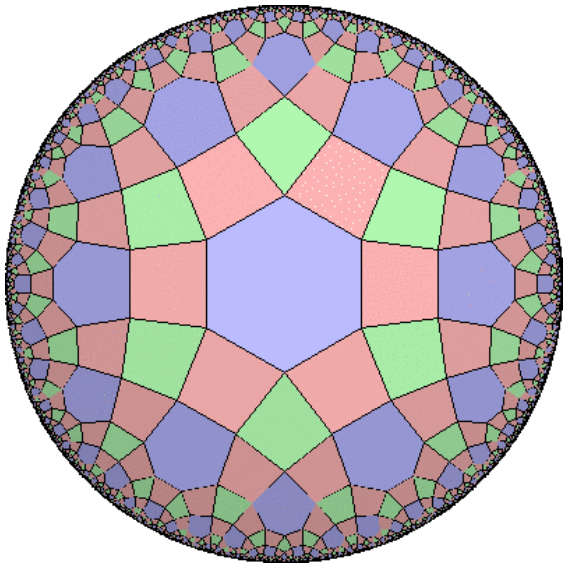

Figure A.13: The Cayley graph of $W_{2,2,2,3}$

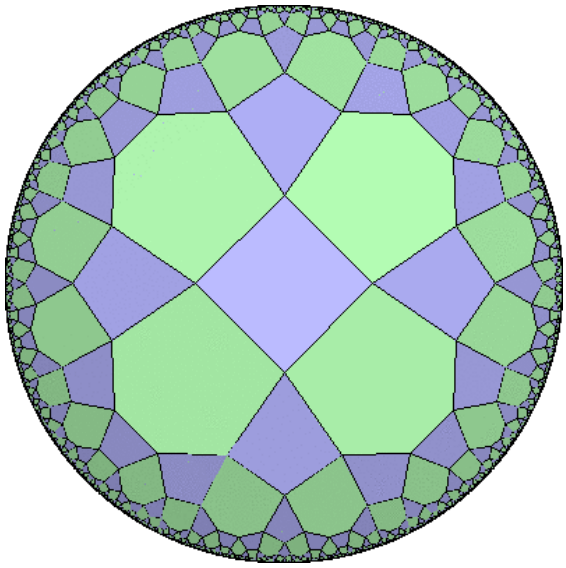

Figure A.14: The Cayley graph of $W_{2,3,2,3}$

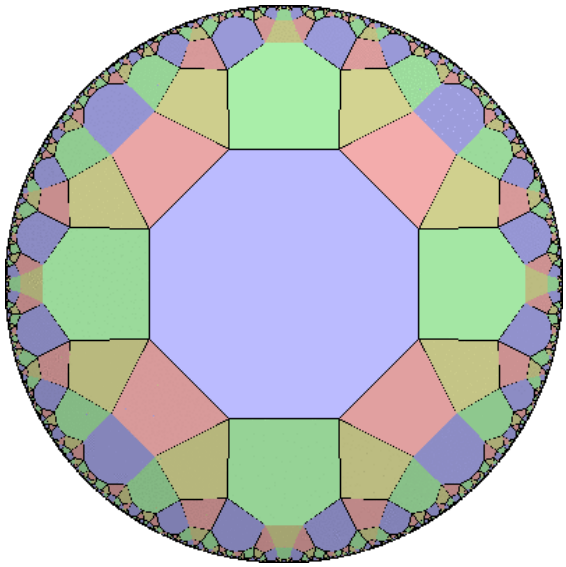

Figure A.15: The Cayley graph of $W_{2,2,3,4}$

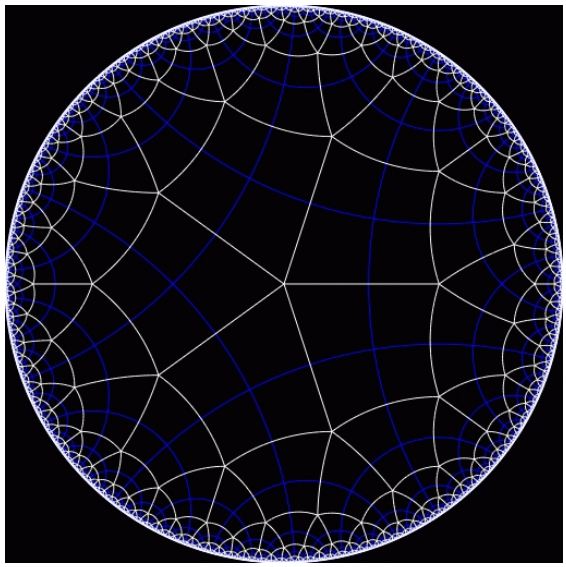

Figure A.16: The Cayley graph of $W_{2,2,2,2,2}$

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